Shahid Chamran University of Ahvaz

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NUMERICAL SOLUTION OF FOKKER-PLANCK-KOLMOGOROV TIME FRACTIONAL DIFFERENTIAL EQUATIONS USING LEGENDRE WAVELET METHOD ALONG WITH CONVERGENCE AND ERROR ANALYSIS

SHABAN MOHAMMADI* AND S. REZA HEJAZI

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Abstract. The aim of this paper is to numerically solve the Fokker-Planck-Kolmogorov fractional-time differential equations using the Legendre wavelet. Also, we analyzed the convergence of function approximation using Legendre wavelets. Introduced the absolute value between the exact answer and the approximate answer obtained by the given numerical methods, and analyzed the error of the numerical method. This method has the advantage of being simple to solve. The results revealed that the suggested numerical method is highly accurate and effective. The results for some numerical examples are documented in table and graph form to elaborate on the efficiency and precision of the suggested method. The simulation was carried out using MATLAB software. In this paper and for the first time, the authors presented results on the numerical simulation for classes of time-fractional differential equations. The authors applied the reproducing Legendre wavelet method for the numerical solutions of nonlinear Fokker-Planck-Kolmogorov time-fractional differential equation. The method presented in the present study can be used by programmers, engineers, and other researchers in this field.

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*Corresponding author.

1. Introduction

Many mathematicians, physicists, and engineers have been interested in differential calculus and fractional integrals in the last decade because fractional-order differential equations better describe and model some of the natural physical processes of dynamic devices than integer-order differential equations. However, after converting natural events to differential equations and systems, it's critical to solve these equations and systems, many of which lack an analytical solution or, if they do, are extremely difficult to locate. As a result, many numerical methods for solving such equations and devices have been presented [3, 18, 19]. The present study aimed to discuss the nonlinear-based Fokker-Planck-Kolmogorov fractional-time differential equations and the wavelet method and the wavelet operation matrix were used for its solution. In the present study, the fractional differential equations are reduced to linear differential equations and the approximate solutions of the primary problem are calculated by introducing the Legendre wavelet and giving the related fractional integral operational matrices. In addition, numerical examples are offered and the numerical findings are evaluated to highlight the simplicity and efficiency of the suggested approaches. One of the most significant subjects in numerical analysis after solving an equation with numerical methods is to analyze the numerical method's convergence to the precise answer and its error analysis to solve the problem. As a result, the convergence of the function approximation is investigated using Legendre wavelets by giving theorems. The absolute value of the error between the precise solution and the approximate response generated from the numerical method is then introduced, and finally, the numerical method's error is analyzed. In the wavelets, there was a significant improvement. Orthogonal bases with desirable features may be established and the difficulties connected to those spaces can be explored by introducing wavelets and their related properties. An integrated framework and coherent theory for several approaches and techniques were provided by the wavelet theory that has been presented and developed sequentially by different scientists in many disciplines [8, 9]. The term "wavelet" refers to a small wave. By applying approximate findings, a wavelet as a conversion is sometimes a tool for showing information, functions, and operators that are better and more efficient than other converters, such as the Fourier transform. Wavelets are used in a variety of fields, including data and image compression, transient tracking, geology and earthquakes, marine and ocean sciences, noise cancellation, pattern recognition, histolysis, fast computing, etc. [5, 20, 17]. Wavelets and wavelet operating matrices have been explored by numerous scientists in the field of numerical analysis, and several issues in the field of numerical solutions related to differential and integral equations have been studied [10, **?**]. Legendre wavelets are defined as a function of the mother wavelet by Legendre polynomials. Wavelets are utilized in a variety of disciplines and applications, as described in earlier chapters. The Legendre wavelet, for example, has been applied to the numerical solution of equations and systems of integral equations [25, 24, 29], initial-boundary value problems [26], differential equations with partial derivatives [11], differential elliptic equations derivatives [31, 28], and fractional differential equations in numerical analysis [1, 4]. In statistical mechanics, the Fokker-Planck equation is a partial differential equation that accounts for the time evolution of the density probability velocity function for the particles influenced by drag and random forces. Brownian motion is described by this equation, which may be extended to include observations expect for velocity [30, 2]. Previous research has suggested a technique for solving two-dimensional Fokker-Planck equations for non-hybrid continuous systems using the finite difference approach, and the proposed method's stability and accuracy have been investigated. Many other articles are written on the numerical solution of Fokker-Planck equations [23, 27, 7]. Fraction calculations are now considered by many researchers. Fractional differential equations are also used in various disciplines such as mechanics, physics, biology and engineering [6, 21]. Due to the increasing application of this group of equations, special attention has been paid to the numerical and precise methods of differential equations. Recently, the use of fractional-order operational matrices to solve fractional-order differential equations has been developed [15, 22]. Many important physical and mechanical problems lead to fractional-order differential equations, but in practice a small number of these equations can be solved analytically and their exact answers obtained. Therefore, we use numerical methods to calculate their approximate answer [16, 13]. Using the Legendre-based operational matrix method to solve differential equations of time fraction and convert it into equations that can be solved with mathematical software is one of the advantages of Legendre wavelet operation method. The biggest advantage of this method is the high speed of convergence [14, 12]. The Legendre wavelet method is utilized in this study to solve the Fokker-Planck-Kolmogorov time-fractional differential equations in the following way [23]:

$$D_t^{\alpha} u - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + \left(\beta - 2\sigma^2\right) x \frac{\partial u}{\partial x} + \left(\beta - \sigma^2\right) u = R\left(x, t\right) \tag{1.1}$$

Initial conditions:

$$\mathrm{u}\left(0, \mathrm{x}\right) = f_0\left(x\right), \quad u_t\left(0, x\right) = f_1\left(x\right), \quad 0 \le x \le 1$$

Boundary conditions:

$$\mathrm{u}\left(t, 0\right) = g_0\left(t\right), \quad u_t\left(t, 1\right) = g_0\left(t\right), \quad 0 \le t \le 1$$

$R(x, t)$ Is the right side function of the equation, which is given for each equation.

2. Preliminieris

2.1. **Riemann-Liouville Integral and fractional derivative.** Suppose that $n > 0$ and fare continuous segments on the interval (α, ∞) and are integrable on any finite sub-interval (α, ∞). Then, the fractional Riemann-Liouville Integral f for $t > a$ of order n is defined as

$$aD_t^{-n} f\left(t\right) = \frac{1}{(n)} \int_{\alpha}^{t} \left(t - T\right)^{n-1} f\left(T\right) dT, \tag{2.1}$$

Which can also be displayed with the symbols $\mathrm{I}_{\mathrm{a}}^{\mathrm{n}}$ or $J_{\mathrm{a}}^{\mathrm{n}}$. In addition, if f is continuous on $[a, t]$, then $\lim_{n \to \alpha} \mathrm{D}_{\mathrm{t}}^{-\mathrm{n}}\ f\left(t\right) = f(t)$. Furthermore, the following equation can be true:

$$aD_t^0 f\left(t\right) = f(t) \tag{2.2}$$

When $n - m \in N$, the definition of (1-1) is compatible $-m$ with -fold integral as follows:

$$\begin{aligned} aD_t^{-m} f\left(t\right) &= \int_{\alpha}^{t} dT_1 \int_{\alpha}^{T_1} dT_2 \ldots \int_{\alpha}^{T_{m-1}} f(T_m) dT_m \\ &= \frac{1}{(m-1)!} \int_{\alpha}^{t} \left(t - T\right)^{m-1} f\left(T\right) dT \qquad m \in N \end{aligned} \tag{2.3}$$

Regarding $m \ge 0$ and $v > -1$, the integral from the defined real order in Equation (2.1) has the following properties:

$$I._{\alpha} D_t^{-n} (t - \alpha)^v = \frac{(v+1)}{(n+v+1)} (t - \alpha)^{n+v} \tag{2.4}$$

$$II. {}_{\alpha}D_t^{-n}k = \frac{k}{(n+1)}(t-\alpha)^n,$$

If $f(t)$ for $t \geq a$ is continuous, then:

$$III. {}_{\alpha}D_t^{-n}({}_{\alpha}D_t^{-m}f(t)) = {}_{\alpha}D_t^{-m}({}_{\alpha}D_t^{-n}f(t)) = {}_{\alpha}D_t^{-n-m}f(t) \tag{2.5}$$

2.2. **Caputo fractional derivative.** Caputo defined a derivative operator in 1976 that differs from previous derivatives in terms of characteristics. The symbol of this operator is as ${}_aD_*^n$ and is defined as

$$\begin{aligned} aD_*^n f(t) &= \frac{1}{(m-n)}\int_{\alpha}^{t}(t-T)^{m-n-1}f^{(m)}(T)\,dT \qquad (m-1 < n \leq m) \\ &= \alpha D_t^{-(m-n)} f^{(m)}(t), \end{aligned} \tag{2.6}$$

On the conditions that $n \to m$ are exercised on the f function, then the Caputo derivative transforms to the mth order derivative of the $f(t)$ function. Suppose that $0 \leq m-1 < n < m$ and function $f(t)$ have $m+1$ continuous bounded derivative in the interval $[a, t]$, then by partial integration for each $t > a$ per $m = 1, 2, \ldots,$ we have:

$$\begin{aligned} \lim_{n \to m_\alpha} D_*^n f(t) &= \lim_{\mathrm{n}\to\mathrm{m}} \Big(\frac{\mathrm{f^{(m)}}(\alpha)(\mathrm{t}-\alpha)^{\mathrm{m-n}}}{(\mathrm{m}-\mathrm{n}+1)} \int_{\alpha}^{\mathrm{t}} (\mathrm{t}-\mathrm{T})^{\mathrm{m-n}}\mathrm{f^{(m+1)}}(\mathrm{T})\,\mathrm{dT}\Big) \\ &= f^{(m)}(\alpha) + \int_{\alpha}^{t} f^{(m+1)}(T)\,dT = f^{(m)}(t). \end{aligned} \tag{2.7}$$

3. Research method

3.1. **Legendre wavelets and Orthogonal system of block-pulse.** In this section, at first Legendre polynomials are defined on the interval $[-1,\ 1]$, then they are translated to the interval $[0,\ 1]$. We define Legendre wavelets based on Legendre polynomials and finally introduce the operational matrix of Legendre wavelets. Legendre polynomials on the interval $[-1,\ 1]$ with the weight function $w(t) = 1$ are defined, recursively, as follows:

$$L_0(t) = 1, \tag{3.1}$$

$$L_1(t) = t, \tag{3.2}$$

$$L_i(t) = \frac{2i-1}{i} t L_{i-1}(t) - \frac{i-1}{i} L_{i-2}(t), \quad i \geq 2 \tag{3.3}$$

According to the Legendre polynomials on the interval $[-1,\ 1]$, it is possible to translate Legendre polynomials on the interval $[0,\ 1]$, by applying changes in a simple variable, and define them in the following way:

$$L_i^*(t) = L_i(2t-1), \qquad i = 0,\ 1,\ 2,\ \ldots \tag{3.4}$$

A set m is a member of the block-pulse functions on the interval $[0.1)$ is defined as follows:

$$b_i(t) = \begin{cases} 1 & \frac{i}{m} \le t \le \frac{i+1}{m} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

Lemma 3.1. *Legendre polynomials are pairwise Orthogonal* [3]. *That is,*

$$\int_0^1 L_i^*(t)\, L_j^*(t)\, dt = \frac{1}{2i+1} = \begin{cases} \frac{1}{2j+1,} & i = j, \\ 0 & i \ne j. \end{cases} \tag{3.6}$$

Legendre wavelets on the interval $[0, 1)$ are defined as follows:

$$\psi_{ij}(\mathrm{t}) = \begin{cases} \sqrt{\frac{2j+1}{2}}\, 2^{\frac{k+1}{2}} L_j\left(2^{k+1}t - (2j+1)\right), & \frac{i}{2^k} \le t < \frac{i+1}{2^k}, \\ 0\ , & \text{otherwiese} \end{cases} \tag{3.7}$$

Where $k, M \in \mathbb{N}$ and $j = 0,\ 1,\ 2,\ \ldots,\ M-1, i = 0,\ 1,\ 2,\ \ldots,\ 2^k - 1$ hold. Sometimes the translated Legendre polynomials are used for the purpose of simplification and ease of calculation, and the Legendre wavelets are defined as follows:

$$\psi_{ij}(\mathrm{t}) = \begin{cases} 2^{k/2}\sqrt{2j+1}L_j^*\left(2^k t - i\right), & \frac{i}{2^k} \le t < \frac{i+1}{2^k}, \\ 0\ , & \text{otherwiese} \end{cases} \tag{3.8}$$

Where $k, M \in \mathbb{N}$ and $= 0,\ 1,\ 2,\ \ldots,\ M-1, i = 0,\ 1,\ 2,\ \ldots,\ 2^k - 1$.
For $k, M \in \mathbb{N}$, insert $m = 2^k M$, and consider the column vector $\psi_m(\mathrm{t})\,,\ m \times 1$ as follows:

$$\begin{aligned} \psi_m(\mathrm{t}) = \big[&\psi_{00}(t), \psi_{01}(t), \ldots, \psi_{0(M-1)}(t),\ \ \psi_{10}(t),\ \psi_{11}(t), \ldots, \\ &\psi_{1(M-1)}(t),\ \ldots,\ \psi_{\left(2^k-1\right)0}(\mathrm{t}),\ \psi_{\left(2^k-1\right)1}(\mathrm{t}),\ \ldots,\ \psi_{\left(2^k-1\right)(M-1)}(t)\big] \end{aligned}$$

Sometimes we insert $n = iM + j + 1$ for simplification of expression in indices of the components of the above vector and rewrite the vector $\psi_{\mathrm{m}}(t)$ as follows:

$$\begin{aligned} \psi_m(t) = \big[&\psi_1,\ \psi_2,\ \ldots,\ \psi_M,\ \psi_{M+1}, \psi_{M+2}, \ldots,\ \psi_{2M}, \ldots, \\ &\psi_{M\left(2^k-1\right)} + 1, \ldots,\ \psi_{M\left(2^k-1\right)} + 2, \ldots,\ \psi_M\big]^T \end{aligned} \tag{3.9}$$

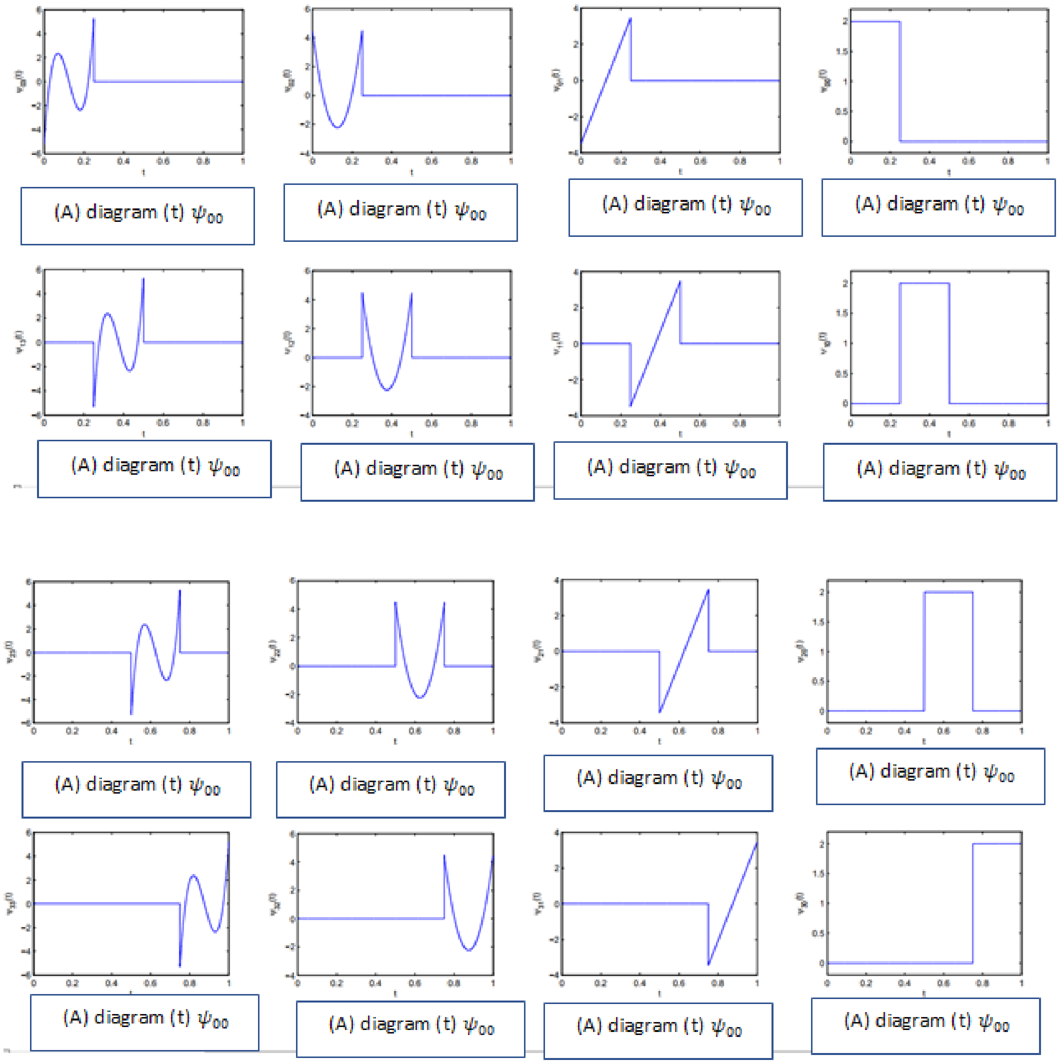

FIGURE 1. Legendre wavelets diagrams for $k = 2$ and $M = 4$

3.2. **Operational matrix of Legendre wavelets.** The collocation meshless local points of $i = 1,\ 2\ldots,\ m,\ t_i$ for $m \in \mathbb{N}$ are defined as follows:

$$t_i = \frac{2i-1}{2m}, \quad i = 1,\ 2,\ \ldots,\ m. \tag{3.10}$$

For $k, M \in \mathbb{N}$, the operational matrix of Legendre wavelets, is a square and invertible matrix and of the order $m = 2^k M$ that is defined as follows [3].

$$\Phi_{m\times m}{=}\left[\Psi_m\left(t_1\right) \quad \Psi_m\left(t_2\right) \quad \Psi_m\left(t_3\right)\ldots \quad \Psi_m\left(t_m\right)\right], \tag{3.11}$$

For example, when $k = 1$ and $M = 4$, the operational matrix of Legendre wavelets $\Phi_{8\times 8}$ will be

$$\Phi_{8\times 8} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\ -1.8371 & -0.6124 & 0.6124 & 1.8371 & 0 & 0 & 0 & 0 \\ 1.0870 & -1.2847 & -1.2847 & 1.0870 & 0 & 0 & 0 & 0 \\ 0.2631 & 1.2570 & -1.2570 & -0.2631 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & 0 & -1.8371 & -0.6124 & -0.6124 & -0.6124 \\ 0 & 0 & 0 & 0 & 1.0870 & -1.2847 & -1.2847 & 1.0870 \\ 0 & 0 & 0 & 0 & 0.2631 & 1.2570 & -1.2570 & -0.2631 \end{bmatrix}$$

and for $k = 2$ *and* $M = 2$, the operational matrix of Legendre wavelets $\Phi_{8\times 8}$ will be

$$\Phi_{8\times 8} = \begin{bmatrix} 2.0000 & 2.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.7321 & 1.7321 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.0000 & 2.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.7321 & 1.7321 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 & 0 \\ 0 & 0 & 0 & 0 & -1.7321 & 1.7321 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0000 & 2.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.7321 & 1.7321 \end{bmatrix}$$

3.3. **Operational matrix of Legendre wavelet fractional integration.** It is necessary to have some appropriate properties in the multiplication of functions by each other and the multiplication of corresponding approximations to solve with the non-linear sentences when intending to solve the equation or system of nonlinear fractional differential equations. Thus, we resort to the properties of block pulse function and their properties , and use them for this purpose. At first, we expand the vector function of the Legendre wavelet according to block pulse functions. The vector function of Legendre wavelet (3.6) can be expanded with the help of the set including m number of the block pulse functions, as follows:

$$\Psi_m(t) = \Phi_{m\times m} B_m\ (t)\,, \tag{3.12}$$

Where: $\mathrm{B}_m(t) = [b_0(t)\,,\ b_1(t)\,,\ b_2(t)\,, \ldots, b_{m-1}(t)]^T$ and $b_i(t)$ are block pulse functions .

Proof. We start from the definition of $\Psi_m(t)$. According to the definition of the block-pulse functions, since: $\psi_m(t) = \sum_{i=0}^{m} \psi_n(t_i)\, b_i(t)$, thus we have

$$\psi_m(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \cdots \\ \psi_m(t) \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{m} \psi_1(t_n)\, b_n(t) \\ \sum_{i=0}^{m} \psi_m(t_n)\, b_n(t) \\ \cdots \\ \sum_{n=1}^{m} \psi_m(t_n)\, b_n(t) \end{bmatrix} = \begin{bmatrix} \psi_1(t_1) & \psi_1(t_1) & \ldots & \psi_1(t_1) \\ \psi_1(t_1) & \psi_1(t_1) & \ldots & \psi_1(t_1) \\ \cdots & \cdots & \cdots & \cdots \\ \psi_1(t_1) & \psi_1(t_1) & \ldots & \psi_1(t_1) \end{bmatrix} \begin{bmatrix} b_1(t) \\ b_2(t) \\ \cdots \\ b_m(t) \end{bmatrix} = \Phi_{m\times m} B_m\ (t)\,.$$

$$(I^a \mathit{\Psi}_m)(t) \approx P^a_{m\times m} \psi_m(t)\,. \tag{3.13}$$

The matrix $P^a_{m\times m}$ is called the Operational matrix of Legendre wavelet fractional integration [3]. we have:

$$(I^a \mathit{\Psi}_m)(t) \approx (I^a \Phi_{m\times m} \mathrm{B}_m)(t) = \Phi_{m\times m} (I^a \mathrm{B}_m)(t) \approx \Phi_{m\times m} F^a \mathrm{B}_m(t)\,. \tag{3.14}$$

Now, we have:

$$P^a_{m\times m} \psi_m(t) = P^a_{m\times m} \Phi_{m\times m} \mathrm{B}_m(t) = \Phi_{m\times m} F^a \mathrm{B}_m(t) \tag{3.15}$$

Therefore, by inserting constant coefficients for $B_m(t)$ in the above equation, the operational matrix of Legendre wavelet fractional integration will be obtained as follows

$$P^a_{m\times m} = \Phi_{m\times m} F^a \Phi^{-1}_{m\times m}, \tag{3.16}$$

□

3.4. **Approximation of functions based on Legendre wavelet.** We approximate the unknown function of the problem according to Legendre wavelets to solve the system fractional differential equations using Legendre wavelets. In this section, we approximate an optional function according to Legendre wavelets. Each optional function of $f(t) \in L^2[0,\ 1)$ can be approximated according to Legendre wavelets. Assume that the expansion of function $f(t)$ according to the Legendre wavelets are as follows:

$$f(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \psi_{ij}(t) \tag{3.17}$$

For$j' \in \mathbb{Z}^{\geq 0}$, optional$i'$, both parts of the equation (3.17) should be multiplied by $\psi_{i'j'}$ (t). then we have

$$\psi_{i'j'}(t)\, f(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \psi_{i'j'}(t)\, \psi_{ij}(t)\,. \tag{3.18}$$

By integrating both parts of the above equation on the interval [0, 1), we have

$$\begin{aligned}\int_0^1 \psi_{i'j'}(t)\, f(t) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathrm{c_{ij}} \psi_{i'j'}(t)\, \psi_{ij}(t)\, \mathrm{dt} = \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \int_0^1 \psi_{i'j'}(t)\, \psi_{ij}(t)\, \mathrm{dt} = c_{i'j}\end{aligned} \tag{3.19}$$

Therefore, the coefficients of c_{ij} for $i, j = 0,\ 1,\ 2,\ \ldots,$ will be obtained as [3].

$$c_{ij} = \langle f(t), \psi_{ij}\ (t) \rangle = \int_0^1 \psi_{ij}(t)\, f(t)\, dt \tag{3.20}$$

Since the calculation of the infinite set is impossible in practice, we approximate it as the following infinite set:

$$f(t) \approx \sum_{i=0}^{2^k-1} \sum_{j=0}^{M-1} c_{ij} \psi_{ij}(\mathrm{t}) = C^T \Psi\ (t) = \hat{f}(t)\,. \tag{3.21}$$

Where:

(3.22)
$$C = \left[c_{00}, c_{01},\ \ldots, c_{0M-1}, c_{10}, c_{11}, \ldots, c_{1M-1}, \ldots, c_{\left(2^k-1\right)0}, c_{\left(2^k-1\right)1}, \ldots, c_{\left(2^k-1\right)M-1}\right]^T$$

(3.23)
$$\Psi(t) = \left[\psi_{00}, \psi_{01}, \ldots, \psi_{0M-1,}, \psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \ldots, \psi_{\left(2^k-1\right)0}, \psi_{\left(2^k-1\right)1}, \ldots, \psi_{\left(2^k-1\right)M-1}\right]^T$$

and $m = 2^k M$. For simplification of expression we place $n = iM + j + 1$ in the indices of the above vector, and write vectors C and $\psi_m(t)$ as follows:

$$C = \left[c_1, c_2,\ \ldots, c_M, c_{M+1}, c_{M+2}, \ldots, c_{2M}, \ldots, c_{M\left(2^k-1\right)+1}, c_{M\left(2^k-1\right)+2}, \ldots, c_m\right]^T \tag{3.24}$$

(3.25)
$$\Psi(t) = \left[\psi_1, \psi_2, \ldots, \psi_{M,}, \psi_{M+1}, \psi_{M+2}, \ldots, \psi_2, \ldots, \psi_{M\left(2^k-1\right)+1}, \psi_{M\left(2^k-1\right)+2}, \ldots, \psi_m\right]^T.$$

The function $f(x,t)$ on the interval $[0,1] \times [0,1]$ can be written using the rabies wavelet as follows:

$$u(t, x\ ) = \sum_{i=0}^{2^{k-1}1} \sum_{j=0}^{M-1}\ \psi_{i,j}(x) {C^T}_{i,j} \Psi(t) \tag{3.26}$$

$$C_{i,j} = (c_{i,1,j,0}, c_{i,1j,1}, \ldots, c_{i,1j,M-1}, \ldots, c_{i,2^{k-1}j,0}, c_{i,2^{k-1}j,1}, \ldots, c_{i,2^{k-1}j,M-1}) \tag{3.27}$$

(3.28)
$$\Psi(t) = \left(\psi_{1,0}(t), \psi_{1,1}(t), \ldots, \psi_{1,M-1}(t), \ldots, \psi_{2^{k-1},0}(t), \psi_{2^{k-1},1}(t), \ldots, \psi_{2^{k-1},M-1}(t)\right)^T$$

Sometimes it may be very complicated or impossible to directly integrate a function and use the equation (3.20). Now, we calculate the coefficients of c_{ij} for $i, j = 0,\ 1,\ 2,\ \ldots,\ m-1$ with the help of the operational matrix of Legendre wavelets and collocation meshless local points. For this purpose, we consider vector $\widehat{f_m}$ as :

$$\widehat{f_m} = \left[\hat{f}(t_1), \hat{f}(t_2), \hat{f}(t_3), \ldots,\ \hat{f}(t_m)\right], \tag{3.29}$$

Where the values presented by t_i for $i = 1, 2, \ldots, m$, are the collocation meshless local points .

Lemma 3.2. *If $\hat{f}(t)$ is the approximation of the function $f(t)$ introduced in equation (3.21), then:*

$$\widehat{f_m} = C^T \Phi_{mm}, \tag{3.30}$$

Thus, the vector of coefficients of Legendre wavelets in $f(t)$ in the equation (3.21) will be

$$C^T = \widehat{f_m} \Phi^{-1}_{m\times m}. \tag{3.31}$$

Theorem 3.3. *According to equation (3.21), we have*

$$\hat{f}(t) = C^T \psi_m\ (t) \tag{3.32}$$

$$= [c_1, c_2,\ \ldots, c_m] \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \cdots \\ \psi_m(t) \end{bmatrix} = c_1\psi_1(t) + c_2\psi_2(t) + \cdots + c_m\psi_m(t). \tag{3.33}$$

Therefore, vector f_m will be as follows:

$$\begin{aligned} \widehat{f_m} &= \left[\hat{f}(t_1), \hat{f}(t_2), \hat{f}(t_3), \ldots,\ \hat{f}(t_m)\right] \\ &= \left[c_1\psi_1(t_1) + \cdots + c_m\psi_m(t_1), \ldots, c_1\psi_1(t_m) + \cdots + +c_m\psi_m(t_m)\right]. \end{aligned} \tag{3.34}$$

Furthermore, according to equation (3.11), we have:

$$C^T\Phi_{mm} = [c_1 \quad c_1 \quad \dots \quad c_m] \begin{bmatrix} \psi_1(t_1) & \psi_1(t_2)\dots & \psi_1(t_m) \\ \psi_2(t_1) & \psi_2(t_2)\dots & \psi_2(t_m) \\ \dots & \dots & \dots \quad \dots \\ \psi_m(t_1) & \psi_m(t_2)\dots & \psi_m(t_m) \end{bmatrix} \tag{3.35}$$

$$= \quad [c_1\psi_1(t_1) + \dots + c_m\psi_m(t_1), \dots, c_1\psi_1(t_m) + \dots + c_m\psi_m(t_m)],$$

Thus, we conclude from the two equations (3.21) and (3.35) that:

$$\widehat{f_m} = C^T\Phi_{mm}. \tag{3.36}$$

Assume that function $f(t)$ has been approximated according to Legendre wavelets in (3.21), then according to equation (3.13), the fractional integration of α order of the function will be calculated as

$$(I^a f)(t) \approx \left(I^a C^T\psi_m\right)(t) = C^T(I^a\psi_m)(t) \approx C^T P^a_{m\times m}\psi_m(t).$$

4. The wavelets method for solving differential equations of Fokker-Planck-Kolmogorov fractional order

For the approximate solution of the Fokker-Planck-Kolmogorov fractional differential equation, the Legendre wavelet method is explained as follows:

$$D_t^\alpha u - \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2} + \left(\beta - 2\sigma^2\right)x\frac{\partial u}{\partial x} + \left(\beta - \sigma^2\right)u = R(x,t) \tag{4.1}$$

Initial conditions:

$$u(0,x) = f_0(x), \quad u_t(0,x) = f_1(x), \qquad 0 \le x \le 1$$

Boundary conditions:

$$u(t,0) = g_0(t), \qquad u_t(t,1) = g_0(t), \qquad 0 \le t \le 1$$

$R(x,t)$ Is the right-side function of the equation given for each equation. Consider:

$$\frac{\partial^4 u(t,x)}{\partial x^2\partial t^2} \approx \Psi^T_{m\times m}(x)C_{m\times m}\Psi_{m\times m}(t) \tag{4.2}$$

By twice integrating with t from both sides of equation (4.2) we have:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx f_0^{''}(x) + tf_1^{''}(x) + \Psi^T_{m\times m}(x)C^m_{m\times m}(mI^2\Psi_{m\times m}m(t)) \tag{4.3}$$

By twice integrating with x from both sides of equation (4.3) we have:

$$\frac{\partial u(t,x)}{\partial x} = \frac{\partial u(t,x)}{\partial x}|_{x=0} + f_0'(x) + f_0'(0) + t(f_1'(x) - f_1'(0)) + (IH^T_{m\times m}(x))^T \Psi_{m\times m}(I^2\Psi_{m\times m}(t)) \tag{4.4}$$

$$u(t,x) \approx u(t,0) + x\frac{\partial u(t,x)}{\partial x}|_{x=0} + (f_0(x) - f_0(0) - x_0'(0)) + t(f_1(x) - f_1(0) - xf_1'(0)) + (I^2\Psi_{m\times m}(x))^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \tag{4.5}$$

Now by applying the boundary conditions and putting $x = 1$, we will have:

$$u(t,1) \approx u(t,0) + x\frac{\partial u(t,x)}{\partial x}|_{x=0} + (f_0(1) - f_0(0) - xf_0'(0)) + t(f_1(1) - f_1(0) - f_1'(0)) + (I^2\Psi_{m\times m}(1))^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \tag{4.6}$$

Therefor:

$$\frac{\partial u(t,x)}{\partial x}|_{x=0} \approx g_1(t) - g_0(t) - (f_0(1) - f_0(0) - f_0'(0)) - t(f_1(1) - f_1(0) - f_1'(0)) - \left(I^2\Psi_{m\times m}(1)\right)^T C_{m\times m}(I^2\Psi_{m\times m}(t)) = K(t) \tag{4.7}$$

Now by placing $K(t)$ in Equation (4.5) we have:

$$u(t,x) \approx g_0(t) + xK(t) + (f_0(x) - f_0(0) - xf_0'(0)) + t(f_1(x) - f_1(0) - xf_1'(0)) + \left(I^2\Psi_{m\times m}(x)\right)^T C_{m\times m}(I^2\Psi_{m\times m}(t)) \tag{4.8}$$

Now we need the fraction derivative $u(t,x)$ according to Equation (4.1). From Equation (4.7) we derive the order fraction α with respect to t:

$$D_t^\alpha u(t,x) \approx D_t^\alpha g_0(t) + xD_t^\alpha K(t) + \left(I^2\Psi_{m\times m}(x)\right)^T C_{m\times m}(I^{2-\alpha}\Psi_{m\times m}(t)) \tag{4.9}$$

And we will have:

$$D_t^\alpha K(t) = D_t^\alpha g_1(t) - D_t^\alpha g_0(t) - \left(I^2\Psi_{m\times m}(1)\right)^T C_{m\times m}(I^{2-\alpha}\Psi_{m\times m}(t)) \tag{4.10}$$

Now convert all approximations ($\approx$) to equals ($=$), and place equations (4.3), (4.4), (4.8) and (4.10) in Equation (4.1), the following linear equation is obtained:

(4.11)

$$D_t^\alpha g_0(t_j) + x_i D_{t\,K}^\alpha(t_j) + \left(I^2 \Psi_{m\times m}(x_i)\right)^T C_{m\times m}(I^{2-\alpha} \Psi_{m\times m}(t)) - \frac{1}{2}\sigma^2 {x_i}^2$$
$$(f''_0(x_i) + t_j f''_1(x_i) + {\Psi^T}_{m\times m}(x_i)\, C_{m\times m}({\Psi^2}_{m\times m}(t_j))) + \left(\beta - 2\sigma^2\right) x_i$$
$$(K(t_j) + f_0'(x_i) - f_0'(0) + t_j(f_1'(x_i) - f_1'(0)) + (I\,\Psi_{m\times m}(x_i))^T C_{m\times m}({\Psi^2}_{m\times m}(t_j)))$$
$$+ \left(\beta - \sigma^2\right)(g_0(t_j) + x_i K(t_j) + (f_0(x_i) - f_0(0) - x_i f_0'(0)) + t_j\big(f_1(x_i)$$
$$- f_1(0) - x_i f_1'(0)\,\big) + \left(I^2 \Psi_{m\times m}(x_i)\right)^T C_{m\times m}(I^2 \Psi_{m\times m}(t_j)) = R(x_i, t_j)$$

5. Convergence and error analysis

In this section, we analyzed the convergence of function approximation using Legendre wavelets by providing some theorems, then we introduced the absolute value between the exact answer and the approximate answer obtained by the given numerical methods, and analyzed the error of the numerical method. The following theorem shows that the function approximation based on Legendre wavelet converges to the function itself, and also an upper bound is obtained for an absolute value of series expansion coefficients.

Theorem 5.1. *Assume that f(t) is a continuous function on the interval* $[0, 1)$ *such that the* $|f''(t)| < R$ *holds. In that case, if the function is expanded to in the form of an infinite set of Legendre wavelets, then the series consistently converge to the function* $f(t)$*. That is,*

$$f(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \psi_{ij}(\mathrm{t}),$$

Then

$$|c_{ij}| < \frac{2\sqrt{3R}}{(2j)^{5/2}\,(2j-3)^2} \tag{5.1}$$

Proof. See [29] for the proof. ∎

To examine and analyze the error of the given numerical method, function approximation using Legendre wavelets, the absolute error between the exact answer and approximate answer will be estimated and defined in the following ways.

If the goal in the numerical method is to find the answer to $u\ (t)$ for the given problem (solving the equation or system fractional differential equations), the exact answer to the problem is shown by $u_{exact}(t)$ and the approximate answer to the problem for $k, m \in \mathbb{Z}^{\geq 0}$ will be shown with $u_{k,M}(t)$, and the absolute error between the approximate and exact answer in t, is calculated using the equation

$$E^{ex}_{k,M}\left(u\left(t\right)\right) = \left|u_{exact}\left(t\right) - u_{k,M}(t)\ \right|, \tag{5.2}$$

However, the answer is often unknown. In this case, we calculate the approximate answer to the problem for the two consecutive values (k, M) and $(k+1, M)$ or (k, M) and $(k, M+1)$ notate as $u_{k,M}(t)$, and $u_{k+1,M}\left(t\right)$ respectively. Thus, the absolute error between the two approximate answers in it for this state is defined as

$$E_{K,M}\left(u\left(t\right)\right) = \left|u_{k,M}\left(t\right) - u_{k+1,M}(t)\ \right|. \tag{5.3}$$

Theorem 5.2. *Assume that $f(t)$ is a continuous function on the interval $[0,1)$ with a bounded second derivative, $|f''(x)| < R$, and $f_{k,M}\left(t\right) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M-1} c_{ij}\psi_{ij}(t)$ provides the approximation of the exact answer, that is $f(t)$, in that case, we have:*

$$\delta_{k,m} < 2\sqrt{3}R\left(\sum_{i=2^k}^{\infty}\frac{1}{\left(2i\right)^5}\sum_{j=M}^{\infty}\left(2j-3\right)^4\right)^{1/2} \tag{5.4}$$

Where

$$\delta_{k,m} = \left(\int_0^1 \left[u^{exact}\left(t\right) - u_{k,M}(t)\right]dt\right)^{1/2} \tag{5.5}$$

Proof. According to the definition of $\delta_{k,m}$ and $f_{k,M}$ in the theorem, we have:

$$\begin{aligned} \delta^2_{k,m} &= \int_0^1 \left[f\left(t\right) - \ c_{ij}\psi_{ij}\left(t\right)\right]^2 dt \\ \delta^2_{k,m} &= \int_0^1 \left[f\left(t\right) - \sum_{i=0}^{2^k-1}\ \sum_{j=0}^{M-1} c_{ij}\psi_{ij}(t)\right]\quad dt \end{aligned} \tag{5.6}$$

$$\int_0^1 \sum_{i=2^k}^{\infty}\ \sum_{j=M}^{\infty} c^2_{ij}(\psi_{ij}(t))^2 dt = \sum_{i=2^k}^{\infty}\ \sum_{j=M}^{\infty} c^2_{ij}2^k\ \ (2j+1)\int_{\frac{i}{2^k}}^{\frac{i+1}{2^k}} L^{*2}_j\left(2^k t - i\right)dt. \tag{5.7}$$

We reuse the change made in the variable in proving the previous theorem is used again, add the $x = 2^k t - i$ and thus $dt = \frac{1}{2^k} dx$. In this case, we have:

$$\delta_{k,m}^2 = \sum_{i=2^k}^{\infty} \sum_{j=M}^{\infty} c_{ij}^2 (2j+1) \int_0^1 L_j^{*2}(x)\, dx = \sum_{i=2^k}^{\infty} \sum_{j=M}^{\infty} c_{ij}^2. \tag{5.8}$$

Thus, we can write:

$$\delta_{k,m}^2 < 12R^2 \left(\sum_{i=2^k}^{\infty} \frac{1}{(2i)^5} \sum_{j=M}^{\infty} \frac{1}{(2j-3)^4} \right), \tag{5.9}$$

And the proof ends. □

As for the case where the exact answer to the problem is unknown, the following theorem indicates the accuracy and convergence of the method.

Theorem 5.3. *Assume $f_{k,M}(t)$ and $f_{K+1,\ \ M}(t)$ are two numerical answers to the unknown function $f(t)$, therefore the estimation of the $\delta_{k,M}$ defined above is convergent for k and M.*

Proof. Considering the definition of $f_{k,M}$ we have:

$$\begin{aligned} E_{K,M}(f) = \ & |f_{k,M} - f_{K+1,\ \ M}| = \ |(f_{k,M} - f) - (f_{k+1,\ \ M} - f)| \\ & \leq |f - f_{k,M}| + |f - f_{k+1,\ \ M}| = E_{k,M}^{ex}(f) + E_{k+1,M}^{ex}(f) \end{aligned}$$

We have:

$$\delta_{k,M} = \ \|E_{K,M}\| E \leq \left\|E_{K,M}^{ex} + E_{K+1,M}^{ex}\right\| E \leq \left\|E_{k,M}^{ex}\right\| E + \left\|E_{k+1,M}^{ex}\right\| E. \tag{5.10}$$

We find out that the norms $\left\|E_{k,M}^{ex}\right\| E$ and $\left\|E_{k+1,M}^{ex}\right\| E$ are convergent, thus the error estimation of $\delta_{k,M}$ is convergent. □

6. Solving numerical examples

Numerical solutions and errors are calculated, evaluated, and provided in tables after evaluating certain numerical instances with conditions of varying initial values. The MATLAB software is used to solve all of the examples.

Example 6.1. In equation (1.1), by placing, $= 1.1$, $\beta = 1$, $\sigma = 0.2$, $m = 3$, $k = 2$, Initial conditions:

$$u(0,x) = 0, \quad u_t(0,x) = 0, \qquad 0 \leq x \leq 1$$

TABLE 1. Example 6.1 error, by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.11*10^{-10}$	$2.18*10^{-10}$	$5.10*10^{-11}$	$6.32*10^{-11}$	$2.08*10^{-11}$
(3.13,3.13)	$4.02*10^{-8}$	$1.84*10^{-8}$	$2.34*10^{-8}$	$5.47*10^{-8}$	$4.06*10^{-8}$
(5.13,5.13)	$3.64*10^{-6}$	$1.52*10^{-6}$	$1.82*10^{-6}$	$4.19*10^{-6}$	$5.28*10^{-6}$
(7.13,7.13)	$2.74*10^{-5}$	$2.01*10^{-5}$	$3.01*10^{-5}$	$2.94*10^{-5}$	$3.64*10^{-5}$
(9.13,9.13)	$3.19*10^{-4}$	$3.84*10^{-4}$	$4.62*10^{-4}$	$6.38*10^{-4}$	$4.15*10^{-4}$
(11.13,11.13)	$1.59*10^{-7}$	$1.04*10^{-7}$	$2.14*10^{-7}$	$3.55*10^{-7}$	$2.07*10^{-7}$

Boundary conditions:

$$u\,(t,0) = 0, \qquad u_t\,(t,1) = 0, \qquad 0 \le t \le 1$$

The right-side functions of the equation:

$$R\,(t,x) = \left(\frac{2t^{2-\alpha}}{\Gamma\,(3-\alpha)} + \frac{1}{2}(\sigma xt\pi)^2 + \left(\beta - \sigma^2\right)t^2\right) sin\,(\pi x) \; + (\beta - 2(\sigma)^2)t^2 x\pi cos(\pi x)$$

The accurate answer of this equation in example 6.1 is $u\,(t,x) = t^2 \sin\,(\pi x)$. Example 6.1 is solved by the Legendre wavelet method for, $\alpha = 1.1$, $\beta = 1$, $\sigma = 0.2$, $m = 3$, $k = 2$ and its error is presented in Tables 1and 2.

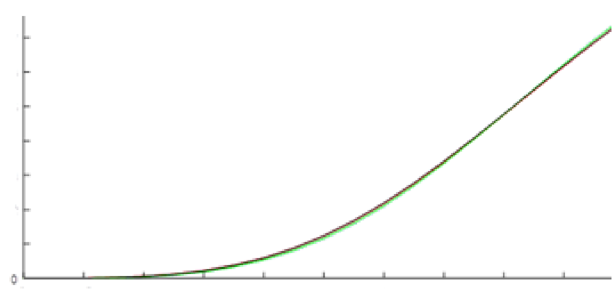

FIGURE 2. Relation of B and error for example 6.1 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

The method of numerical solution for $\alpha = 1.1$, $\beta = 1$, $\sigma = 0.2$, $m = 3$, $k = 2$is presented in Table 3 , 4.

TABLE 2. Example 6.1 error, by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 4}$, $\boldsymbol{k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.02*10^{-10}$	$2.02*10^{-10}$	$5.02*10^{-11}$	$6.10*10^{-11}$	$2.01*10^{-11}$
(3.13,3.13)	$4.00*10^{-8}$	$1.77*10^{-8}$	$2.22*10^{-8}$	$4.19*10^{-8}$	$4.00*10^{-8}$
(5.13,5.13)	$3.52*10^{-6}$	$1.48*10^{-6}$	$1.67*10^{-6}$	$4.10*10^{-6}$	$5.12*10^{-6}$
(7.13,7.13)	$2.10*10^{-5}$	$2.00*10^{-5}$	$2.88*10^{-5}$	$2.45*10^{-5}$	$3.23*10^{-5}$
(9.13,9.13)	$3.08*10^{-4}$	$3.66*10^{-4}$	$4.38*10^{-4}$	$6.05*10^{-4}$	$3.82*10^{-4}$
(11.13,11.13)	$1.51*10^{-7}$	$1.02*10^{-7}$	$2.11*10^{-7}$	$3.34*10^{-7}$	$2.00*10^{-7}$

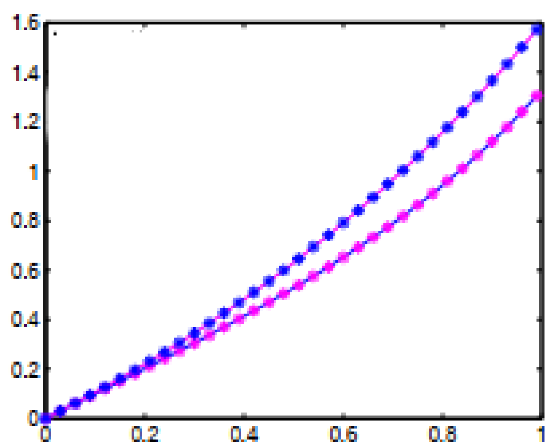

FIGURE 3. Approximate and exact solution, respectively for example 6.1 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

Example 6.2. The numerical solution of the following equation:
In equation (1.1), by placing $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$
Initial conditions:

$$u\,(0,x) = 0, \quad u_t\,(0,x) = 0, \qquad 0 \le x \le 1$$

Boundary conditions:

$$u\,(t,0) = \mathrm{t}^3, \qquad u_t\,(t,1) = et^3, \qquad 0 \le t \le 1$$

TABLE 3. The numerical solution of example 6.1 by placing, $\boldsymbol{\alpha = 1.1\ ,\ \beta = 1\ ,\ \sigma = 0.2\ ,\ m = 3\ ,\ k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$2.55*10^{-4}$	$2.54*10^{-4}$	$2.54*10^{-4}$	$2.53*10^{-4}$	$2.53*10^{-4}$
(3.13,3.13)	$3.84*10^{-3}$	$3.83*10^{-3}$	$3.83*10^{-3}$	$3.82*10^{-3}$	$3.82*10^{-3}$
(5.13,5.13)	$1.64*10^{-2}$	$1.63*10^{-2}$	$1.63*10^{-2}$	$1.62*10^{-2}$	$1.62*10^{-2}$
(7.13,7.13)	$1.79*10^{-2}$	$1.78*10^{-2}$	$1.78*10^{-2}$	$1.76*10^{-2}$	$1.76*10^{-2}$
(9.13,9.13)	$2.13*10^{-2}$	$2.12*10^{-2}$	$2.12*10^{-2}$	$2.11*10^{-2}$	$2.11*10^{-2}$
(11.13,11.13)	$3.18*10^{-2}$	$3.18*10^{-2}$	$3.17*10^{-2}$	$3.17*10^{-2}$	$3.16*10^{-2}$

TABLE 4. The numerical solution of example 6.1 by placing, $\boldsymbol{\alpha = 1.1\ ,\ \beta = 1\ ,\ \sigma = 0.2\ ,\ m = 4\ ,\ k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$2.87*10^{-4}$	$2.41*10^{-4}$	$2.88*10^{-4}$	$2.36*10^{-4}$	$2.66*10^{-4}$
(3.13,3.13)	$3.92*10^{-3}$	$3.64*10^{-3}$	$3.90*10^{-3}$	$3.67*10^{-3}$	$3.74*10^{-3}$
(5.13,5.13)	$1.71*10^{-2}$	$1.60*10^{-2}$	$1.75*10^{-2}$	$1.54*10^{-2}$	$1.53*10^{-2}$
(7.13,7.13)	$1.66*10^{-2}$	$1.55*10^{-2}$	$1.83*10^{-2}$	$1.58*10^{-2}$	$1.62*10^{-2}$
(9.13,9.13)	$2.10*10^{-2}$	$2.01*10^{-2}$	$2.01*10^{-2}$	$2.00*10^{-2}$	$2.00*10^{-2}$
(11.13,11.13)	$3.08*10^{-2}$	$3.03*10^{-2}$	$3.11*10^{-2}$	$3.03*10^{-2}$	$3.08*10^{-2}$

The right-side functions of the equation:

$$R\left(t,x\right)=\left(\frac{\Gamma\left(4\right)}{\Gamma\left(4-\alpha\right)}t^{3-\alpha}-\frac{1}{2}\sigma^{2}x^{2}t^{3}+(\beta-2\sigma^{2})xt^{3}+(\beta-\sigma^{2})t^{3}\right)e^{x}$$

The accurate response to this equation in example 6.2 is$u\left(t,x\right)=t^{3}e^{x}$. Example 6.2 is solved by the Legendre wavelet method for $\boldsymbol{\alpha = 1.1\ ,\ \beta = 0.5\ ,\ \sigma = 0.2\ ,\ m = 3\ ,\ k = 2}$ and its error has been shown in Table 7 , 8.

TABLE 5. The exact solution of example 6.1 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	2.44*10^{-4}	2.43*10^{-4}	2.43*10^{-4}	2.42*10^{-4}	2.42*10^{-4}
(3.13,3.13)	3.80*10^{-3}	3.80*10^{-3}	3.80*10^{-3}	3.81*10^{-3}	3.81*10^{-3}
(5.13,5.13)	1.59*10^{-2}	1.59*10^{-2}	1.59*10^{-2}	1.58*10^{-2}	1.57*10^{-2}
(7.13,7.13)	1.76*10^{-2}	1.75*10^{-2}	1.75*10^{-2}	1.74*10^{-2}	1.73*10^{-2}
(9.13,9.13)	2.19*10^{-2}	2.18*10^{-2}	2.15*10^{-2}	2.14*10^{-2}	2.14*10^{-2}
(11.13,11.13)	3.12*10^{-2}	3.12*10^{-2}	3.11*10^{-2}	3.11*10^{-2}	3.10*10^{-2}

TABLE 6. The exact solution of example 6.1 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 1}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 4}$, $\boldsymbol{k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	2.41*10^{-4}	2.41*10^{-4}	2.40*10^{-4}	2.41*10^{-4}	2.40*10^{-4}
(3.13,3.13)	3.77*10^{-3}	3.77*10^{-3}	3.77*10^{-3}	3.78*10^{-3}	3.78*10^{-3}
(5.13,5.13)	1.54*10^{-2}	1.55*10^{-2}	1.55*10^{-2}	1.53*10^{-2}	1.52*10^{-2}
(7.13,7.13)	1.71*10^{-2}	1.69*10^{-2}	1.71*10^{-2}	1.69*10^{-2}	1.71*10^{-2}
(9.13,9.13)	2.03*10^{-2}	2.04*10^{-2}	2.05*10^{-2}	2.05*10^{-2}	2.04*10^{-2}
(11.13,11.13)	3.18*10^{-2}	3.18*10^{-2}	3.17*10^{-2}	3.17*10^{-2}	3.16*10^{-2}

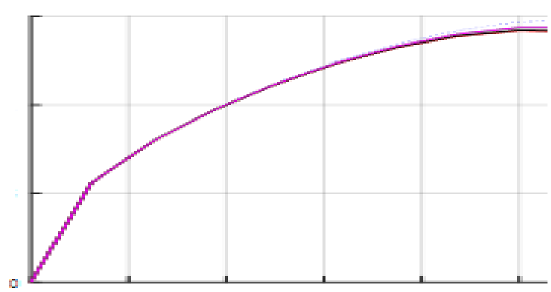

FIGURE 4. Relation of B and error for example 6.2 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

TABLE 7. The error of example 6.2 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	4.52*10^{-8}	5.41*10^{-8}	5.19*10^{-8}	4.99*10^{-8}	5.55*10^{-8}
(3.13,3.13)	3.76*10^{-6}	4.12*10^{-6}	4.63*10^{-7}	4.47*10^{-6}	4.34*10^{-6}
(5.13,5.13)	2.91*10^{-9}	3.04*10^{-9}	5.33*10^{-8}	3.12*10^{-8}	3.17*10^{-8}
(7.13,7.13)	2.74*10^{-4}	2.14*10^{-4}	3.55*10^{-4}	2.39*10^{-4}	2.37*10^{-4}
(9.13,9.13)	3.21*10^{-4}	3.01*10^{-4}	4.12*10^{-4}	4.19*10^{-4}	4.20*10^{-4}
(11.13,11.13)	1.18*10^{-3}	1.44*10^{-3}	2.01*10^{-3}	1.67*10^{-3}	1.33*10^{-3}

TABLE 8. The error of example 6.2 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 4}$, $\boldsymbol{k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	4.33*10^{-8}	5.35*10^{-8}	5.10*10^{-8}	4.88*10^{-8}	5.34*10^{-8}
(3.13,3.13)	3.66*10^{-6}	4.10*10^{-6}	4.52*10^{-7}	4.16*10^{-6}	4.23*10^{-6}
(5.13,5.13)	2.80*10^{-9}	3.00*10^{-9}	5.12*10^{-8}	3.10*10^{-8}	3.01*10^{-8}
(7.13,7.13)	2.63*10^{-4}	2.23*10^{-4}	3.47*10^{-4}	2.23*10^{-4}	2.31*10^{-4}
(9.13,9.13)	3.10*10^{-4}	3.08*10^{-4}	4.01*10^{-4}	4.25*10^{-4}	4.18*10^{-4}
(11.13,11.13)	1.07*10^{-3}	1.39*10^{-3}	2.01*10^{-3}	1.60*10^{-3}	1.27*10^{-3}

In Table 9, 9, the numerical solution method for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$ has been shown.

Example 6.3. The numerical solution of the following equation:
In equation (1.1), by placing , $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$

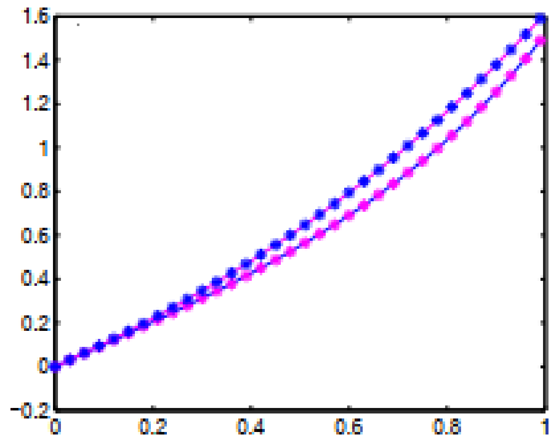

FIGURE 5. Approximate and exact solution, respectively for example 6.2 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

TABLE 9. The numerical solution of example 6.2 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$5.55{*}10^{-4}$	$5.54{*}10^{-4}$	$5.53{*}10^{-4}$	$5.52{*}10^{-4}$	$5.55{*}10^{-4}$
(3.13,3.13)	$2.84{*}10^{-3}$	$2.83{*}10^{-3}$	$2.82{*}10^{-3}$	$2.82{*}10^{-3}$	$2.81{*}10^{-3}$
(5.13,5.13)	$1.55{*}10^{-2}$	$1.54{*}10^{-2}$	$1.53{*}10^{-2}$	$1.53{*}10^{-2}$	$1.51{*}10^{-2}$
(7.13,7.13)	$3.28{*}10^{-2}$	$3.27{*}10^{-2}$	$3.26{*}10^{-2}$	$3.25{*}10^{-2}$	$3.25{*}10^{-2}$
(9.13,9.13)	$1.13{*}10^{-2}$	$1.12{*}10^{-2}$	$1.11{*}10^{-2}$	$1.11{*}10^{-2}$	$1.10{*}10^{-2}$
(11.13,11.13)	$7.18{*}10^{-2}$	$7.17{*}10^{-2}$	$7.16{*}10^{-2}$	$3.16{*}10^{-2}$	$3.15{*}10^{-2}$

Initial conditions:

$$u\,(0,x) = 0, \quad u_t\,(0,x) = 0, \qquad 0 \le x \le 1$$

Boundary conditions:

$$u\,(t,0) = 0, \qquad u_t\,(t,1) = t^3 sin^2 x \ , \qquad 0 \le t \le 1$$

The right-side functions of the equation:

$$R\,(t,x) = (\frac{\Gamma\,(4)}{\Gamma\,(4-\alpha)} t^{3-\alpha} + (\beta - \sigma^2) t^3) sin^2 x - \sigma^2 x^2 t^3 cos(2x) + (\beta - \sigma^2) x t^3 sin(2x)$$

TABLE 10. The numerical solution of example 6.2 by placing, $\boldsymbol{\alpha} = \mathbf{1.1}$, $\boldsymbol{\beta} = \mathbf{0.5}$, $\boldsymbol{\sigma} = \mathbf{0.2}$, $\boldsymbol{m} = \mathbf{4}$, $\boldsymbol{k} = \mathbf{3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.16*10^{-4}	5.18*10^{-4}	5.17*10^{-4}	5.17*10^{-4}	5.22*10^{-4}
(3.13,3.13)	2.66*10^{-3}	2.66*10^{-3}	2.64*10^{-3}	2.63*10^{-3}	2.64*10^{-3}
(5.13,5.13)	1.47*10^{-2}	1.46*10^{-2}	1.46*10^{-2}	1.47*10^{-2}	1.45*10^{-2}
(7.13,7.13)	3.33*10^{-2}	3.32*10^{-2}	3.31*10^{-2}	3.30*10^{-2}	3.30*10^{-2}
(9.13,9.13)	1.48*10^{-2}	1.47*10^{-2}	1.46*10^{-2}	1.46*10^{-2}	1.45*10^{-2}
(11.13,11.13)	7.30*10^{-2}	7.29*10^{-2}	7.28*10^{-2}	3.27*10^{-2}	3.26*10^{-2}

TABLE 11. The exact solution of example 6.2 by placing, $\boldsymbol{\alpha} = \mathbf{1.1}$, $\boldsymbol{\beta} = \mathbf{0.5}$, $\boldsymbol{\sigma} = \mathbf{0.2}$, $\boldsymbol{m} = \mathbf{3}$, $\boldsymbol{k} = \mathbf{2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.12*10^{-4}	5.12*10^{-4}	5.11*10^{-4}	5.11*10^{-4}	5.12*10^{-4}
(3.13,3.13)	2.42*10^{-3}	2.43*10^{-3}	2.43*10^{-3}	2.42*10^{-3}	2.42*10^{-3}
(5.13,5.13)	1.32*10^{-2}	1.33*10^{-2}	1.33*10^{-2}	1.32*10^{-2}	1.31*10^{-2}
(7.13,7.13)	3.22*10^{-2}	3.21*10^{-2}	3.20*10^{-2}	3.20*10^{-2}	3.21*10^{-2}
(9.13,9.13)	1.32*10^{-2}	1.30*10^{-2}	1.30*10^{-2}	1.30*10^{-2}	1.29*10^{-2}
(11.13,11.13)	7.08*10^{-2}	7.07*10^{-2}	7.06*10^{-2}	3.06*10^{-2}	3.05*10^{-2}

The accurate response to this equation in example 6.3 is$u\,(t,x) = t^3 sin^2 x$. Example 6.3 is solved by the Legendre wavelet method for $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$ and its error has been shown in Tables 13 , 14.

TABLE 12. The exact solution of example 6.2 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 4}$, $\boldsymbol{k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.19*10^{-4}	5.18*10^{-4}	5.19*10^{-4}	5.18*10^{-4}	5.19*10^{-4}
(3.13,3.13)	2.49*10^{-3}	2.50*10^{-3}	2.48*10^{-3}	2.48*10^{-3}	2.49*10^{-3}
(5.13,5.13)	1.44*10^{-2}	1.44*10^{-2}	1.40*10^{-2}	1.43*10^{-2}	1.43*10^{-2}
(7.13,7.13)	3.21*10^{-2}	3.22*10^{-2}	3.20*10^{-2}	3.20*10^{-2}	3.20*10^{-2}
(9.13,9.13)	1.45*10^{-2}	1.45*10^{-2}	1.45*10^{-2}	1.46*10^{-2}	1.46*10^{-2}
(11.13,11.13)	7.22*10^{-2}	7.25*10^{-2}	7.25*10^{-2}	3.22*10^{-2}	3.24*10^{-2}

TABLE 13. The error of example 6.3 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	3.52*10^{-8}	5.85*10^{-8}	5.19*10^{-8}	4.84*10^{-8}	5.60*10^{-8}
(3.13,3.13)	3.18*10^{-6}	4.24*10^{-6}	4.63*10^{-7}	4.55*10^{-6}	4.23*10^{-6}
(5.13,5.13)	2.91*10^{-9}	3.11*10^{-9}	5.02*10^{-8}	3.11*10^{-8}	3.27*10^{-8}
(7.13,7.13)	2.45*10^{-4}	2.20*10^{-4}	3.44*10^{-4}	2.44*10^{-4}	2.21*10^{-4}
(9.13,9.13)	3.39*10^{-4}	3.21*10^{-4}	4.19*10^{-4}	4.45*10^{-4}	4.37*10^{-4}
(11.13,11.13)	1.71*10^{-3}	1.59*10^{-3}	2.11*10^{-3}	1.55*10^{-3}	1.47*10^{-3}

In Table 15 , 16 the numerical solution method for $\alpha = 1.1$, $\beta = 0.5$, $\sigma = 0.2$, $m = 3$, $k = 2$ has been shown.

TABLE 14. The error of example 6.3 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 4}$, $\boldsymbol{k = 3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	$3.71{*}10^{-8}$	$5.75{*}10^{-8}$	$5.55{*}10^{-8}$	$4.72{*}10^{-8}$	$5.55{*}10^{-8}$
(3.13,3.13)	$3.32{*}10^{-6}$	$4.10{*}10^{-6}$	$4.24{*}10^{-7}$	$4.23{*}10^{-6}$	$4.10{*}10^{-6}$
(5.13,5.13)	$2.77{*}10^{-9}$	$3.58{*}10^{-9}$	$5.10{*}10^{-8}$	$3.47{*}10^{-8}$	$3.33{*}10^{-8}$
(7.13,7.13)	$2.44{*}10^{-4}$	$2.47{*}10^{-4}$	$3.23{*}10^{-4}$	$2.32{*}10^{-4}$	$2.30{*}10^{-4}$
(9.13,9.13)	$3.41{*}10^{-4}$	$3.45{*}10^{-4}$	$4.10{*}10^{-4}$	$4.21{*}10^{-4}$	$4.33{*}10^{-4}$
(11.13,11.13)	$1.85{*}10^{-3}$	$1.69{*}10^{-3}$	$2.23{*}10^{-3}$	$1.63{*}10^{-3}$	$1.59{*}10^{-3}$

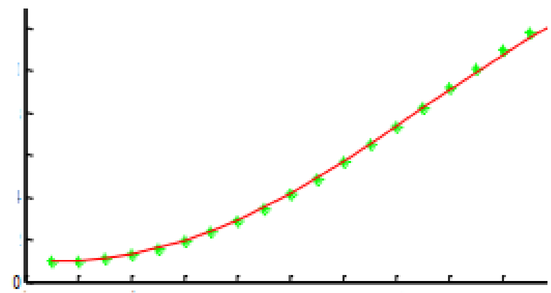

FIGURE 6. Relation of B and error for example 6.3 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

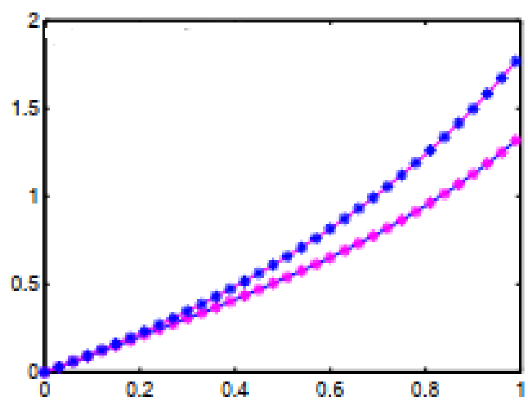

FIGURE 7. Approximate and exact solution, respectively for example 6.3 for $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

TABLE 15. The numerical solution of example 6.3 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.47*10^{-4}	5.42*10^{-4}	5.77*10^{-4}	5.78*10^{-4}	5.44*10^{-4}
(3.13,3.13)	2.75*10^{-3}	2.76*10^{-3}	2.75*10^{-3}	2.74*10^{-3}	2.79*10^{-3}
(5.13,5.13)	1.62*10^{-2}	1.67*10^{-2}	1.54*10^{-2}	1.72*10^{-2}	1.47*10^{-2}
(7.13,7.13)	3.31*10^{-2}	3.33*10^{-2}	3.36*10^{-2}	3.36*10^{-2}	3.33*10^{-2}
(9.13,9.13)	1.19*10^{-2}	1.28*10^{-2}	1.01*10^{-2}	1.22*10^{-2}	1.19*10^{-2}
(11.13,11.13)	7.63*10^{-2}	7.37*10^{-2}	7.42*10^{-2}	3.43*10^{-2}	3.01*10^{-2}

TABLE 16. The exact solution of example 6.2 by placing, $\boldsymbol{\alpha = 1.1}$, $\boldsymbol{\beta = 0.5}$, $\boldsymbol{\sigma = 0.2}$, $\boldsymbol{m = 3}$, $\boldsymbol{k = 2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.88*10^{-4}	5.85*10^{-4}	5.80*10^{-4}	5.82*10^{-4}	5.84*10^{-4}
(3.13,3.13)	2.64*10^{-3}	2.69*10^{-3}	2.64*10^{-3}	2.65*10^{-3}	2.66*10^{-3}
(5.13,5.13)	1.74*10^{-2}	1.75*10^{-2}	1.76*10^{-2}	1.74*10^{-2}	1.73*10^{-2}
(7.13,7.13)	3.49*10^{-2}	3.49*10^{-2}	3.46*10^{-2}	3.44*10^{-2}	3.44*10^{-2}
(9.13,9.13)	1.23*10^{-2}	1.25*10^{-2}	1.20*10^{-2}	1.23*10^{-2}	1.34*10^{-2}
(11.13,11.13)	7.55*10^{-2}	7.50*10^{-2}	7.49*10^{-2}	3.45*10^{-2}	3.44*10^{-2}

7. DISCUSSION AND CONCLUSION

Wavelets are one of the useful tools for numerical analysis of equations. The Legendre wavelet is used to solve fractional differential equations and systems. The Legendre wavelets have been introduced in this study. To do this, Legendre polynomials were initially developed, followed by the identification of Legendre wavelets and Legendre wavelet operating matrices, and finally, the method of approximation of functions in terms of Legendre wavelets. The Fokker-Planck-Kolmogorov fractional differential equations were

TABLE 17. The numerical solution of example 6.3 by placing, $\boldsymbol{\alpha} = \mathbf{1.1}$, $\boldsymbol{\beta} = \mathbf{0.5}$, $\boldsymbol{\sigma} = \mathbf{0.2}$, $\boldsymbol{m} = \mathbf{3}$, $\boldsymbol{k} = \mathbf{2}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.40*10^{-4}	5.40*10^{-4}	5.42*10^{-4}	5.43*10^{-4}	5.44*10^{-4}
(3.13,3.13)	2.68*10^{-3}	2.69*10^{-3}	2.68*10^{-3}	2.69*10^{-3}	2.68*10^{-3}
(5.13,5.13)	1.55*10^{-2}	1.52*10^{-2}	1.52*10^{-2}	1.55*10^{-2}	1.54*10^{-2}
(7.13,7.13)	3.09*10^{-2}	3.10*10^{-2}	3.10*10^{-2}	3.10*10^{-2}	3.10*10^{-2}
(9.13,9.13)	1.25*10^{-2}	1.24*10^{-2}	1.24*10^{-2}	1.25*10^{-2}	1.25*10^{-2}
(11.13,11.13)	7.80*10^{-2}	7.82*10^{-2}	7.82*10^{-2}	3.83*10^{-2}	3.80*10^{-2}

TABLE 18. The numerical solution of example 6.3 by placing, $\boldsymbol{\alpha} = \mathbf{1.1}$, $\boldsymbol{\beta} = \mathbf{0.5}$, $\boldsymbol{\sigma} = \mathbf{0.2}$, $\boldsymbol{m} = \mathbf{4}$, $\boldsymbol{k} = \mathbf{3}$

(x,t)	α=1.1	α=1.3	α=1.5	α=1.7	α=1.9
(1.13,1.13)	5.36*10^{-4}	5.37*10^{-4}	5.35*10^{-4}	5.36*10^{-4}	5.37*10^{-4}
(3.13,3.13)	2.52*10^{-3}	2.53*10^{-3}	2.24*10^{-3}	2.23*10^{-3}	2.25*10^{-3}
(5.13,5.13)	1.61*10^{-2}	1.62*10^{-2}	1.63*10^{-2}	1.62*10^{-2}	1.62*10^{-2}
(7.13,7.13)	3.22*10^{-2}	3.24*10^{-2}	3.23*10^{-2}	3.24*10^{-2}	3.25*10^{-2}
(9.13,9.13)	1.19*10^{-2}	1.20*10^{-2}	1.20*10^{-2}	1.19*10^{-2}	1.21*10^{-2}
(11.13,11.13)	7.44*10^{-2}	7.44*10^{-2}	7.45*10^{-2}	3.34*10^{-2}	3.35*10^{-2}

solved using the Legendre wavelet fractional integral operational matrix. The numerical approach was provided with a theorem after solving the aforementioned equation, and the convergence analysis of the function approximation was performed using the Legendre wavelet. The absolute value of the error between the precise and approximate answers provided by the numerical technique was then introduced, and the numerical method's error was analyzed.

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Shaban Mohammadi
Department of Mathematics,
Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran
Email: arashmoh2019@gmail.com

Seyed Reza Hejazi
Department of Mathematics,
Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran
Email: ra.hejazi@gmail.com