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EXACT SOLUTIONS OF (3+1)-DIMENSIONAL WAVE EQUATION VIA LIE BRACKETS OF SYMMETRIES

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ABSTRACT. In this paper, the Lie symmetry method and Lie brackets of vector fields are used in order to find some new solutions of (3+1)-dimensional sourceless wave equation. The obtained solutions are classified to two categories; polynomial and non-polynomial exact solutions. Because of the properties of the Lie brackets and the symmetries, a generalized method is implemented for constructing new solutions from old solutions. We demonstrate the generation of such polynomial and non-polynomial solutions through the medium of the group theoretical properties of the equation. It is noteworthy that this method could be used when the equations have two special kinds of symmetries which will be mentioned below.

1. Introduction

One of the most famous and applicable hyperbolic PDE is the second non-linear $(n+1)$-dimensional wave equation with a source term

$$u_{tt} = \sum_{i=1}^{n} (f_i(u) u_{x^i})_{x^i} + g(u), \tag{1.1}$$

which is using for the description of waves as they occur in physics such as sound waves, light waves and water waves. Another usage of Eq. (1.1) is in acoustics, electromagnetics and fluid dynamics. The quantity u may be, for example, the pressure in a liquid or gas, or

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the displacement, along some specific direction, of the particles of a vibrating solid away from their resting positions. In this paper a special case of the Eq. (1.1) is considered as follows

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}, \tag{1.2}$$

then the wide range of solutions with Lie symmetry method are given. This method is based on the Lie brackets of the symmetries and their relations for finding exact solutions of differential equations. Symmetries of differential equations can reduce the primary equation to a simpler form. Then, the solutions of the reduced form are called the similarity solutions. In the literature, one can find the classical reduction process [1], and the moving frame-based reduction process [7, 8, 9, 10]. Also symmetry groups can be used for classifying different symmetry classes of solutions. A similar work is done for (2+1)-dimensional wave equation [11].

As we will see, the Eq. (1.2) admits $16 + \infty$ infinitesimal generators. The classical solutions are recovered with the use of non-generic symmetries to construct similar solutions. Further solutions, both polynomial and non-polynomial, are constructed by using the invariant of the Lie point symmetries as seed solutions and the property of mapping solutions into solutions. These solutions are analogous to the well-known wave polynomials.

The paper is outlined in three sections, including a conclusion. Lie point symmetries of the Eq. (1.2) including the invariant for finding the similarity solution are given in the second section. Then, we applied the Lie bracket of the symmetries to find some new solutions from the old solutions in the third section. Finally, some special solutions are plotted at the end of this section.

2. Lie symmetries of the Eq. (1.2)

The method of finding symmetries of differential equations is a routine and standard procedure. There are still many authors using this method to find the exact solutions [1, 6, 16, 17] of non-linear differential equations. The general procedure to obtain Lie symmetries of differential equations, and their applications for finding analytic solutions of the equations are described in detail in several monographs on the subject (e.g. [2, 3, 8]) and in numerous papers in the literature (e.g. [4, 6, 13, 14, 15, 16, 18]). Recently, the extended Lie symmetry method to fractional differential equations is far interesting and is used in so many articles [5, 12, 19, 20].

The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of

$$X = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial y} + \xi_4\frac{\partial}{\partial z} + \phi\frac{\partial}{\partial u}. \tag{2.1}$$

Applying the second prolongation of the vector field (2.1) on (1.2) yields a sixteen dimensional Lie algebra of symmetries spanned by the following infinitesimal generators:

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = \frac{\partial}{\partial x}, \qquad X_5 = t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t},\\
&X_6 = t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}, \qquad X_7 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \qquad X_8 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\\
&X_9 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad X_{10} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X_{11} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
&X_{12} = xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + 1/2(z^2 + t^2 - x^2 - y^2\frac{\partial}{\partial z} + tz\frac{\partial}{\partial t} - uz\frac{\partial}{\partial u},\\
&X_{13} = xy\frac{\partial}{\partial x} + zy\frac{\partial}{\partial z} + \frac{1}{2}(y^2 + t^2 - x^2 - z^2\frac{\partial}{\partial z} + ty\frac{\partial}{\partial t} - uy\frac{\partial}{\partial u},\\
&X_{14} = xt\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + \frac{1}{2}(z^2 + t^2 + x^2 + y^2)\frac{\partial}{\partial t} + tz\frac{\partial}{\partial z} - ut\frac{\partial}{\partial u},\\
&X_{15} = -xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z} + \frac{1}{2}(z^2 - t^2 - x^2 + y^2)\frac{\partial}{\partial x} - tx\frac{\partial}{\partial t} + ux\frac{\partial}{\partial u},\\
&X_{16} = u\frac{\partial}{\partial u}.
\end{aligned}$$

including a pseudo Lie algebra spanned by the vector field

$$X_\infty = \alpha(x, y, z, t)\frac{\partial}{\partial u},$$

where $\alpha(x, y, z, t)$ is a solution of Eq. (1.2).

The structure of the Lie algebra of the symmetries is coming from their origination. Both operators X_{16} and X_∞ together, is a feature of linear evolution equation and those non-linear evolution equation which can be linearized by means of a point transformation. For a linear evolution equation the function, $\alpha(x, y, z, t)$, is a solution of the equation itself as is the case of the wave Eq. (1.2).

An important preponderance of symmetry group method is to construct new solutions from known solutions. To do this, the infinitesimals are considered and their corresponding invariants should be determined. This is a standard method to be found in many texts.

But, readers are reminded that X_{16} and X_∞ do not provide similarity solutions. As a reminder, some examples about the method of constructing similarity solutions are provided here.

For example, the associated Lagrange's system for X_1 is

$$\frac{dy}{d\varepsilon} = 1, \quad \frac{dx}{d\varepsilon} = \frac{dz}{d\varepsilon} = \frac{dt}{d\varepsilon} = \frac{du}{d\varepsilon} = 0, \tag{2.2}$$

where ε is a parameter.

The solution of the system (2.2) yields invariants x, z, t and $u = g(t, x, z)$. Inserting these new variables to Eq. (1.2), concludes that

$$g_{tt} - g_{xx} - g_{zz} = 0. \tag{2.3}$$

The Eq. (2.3) has two different solutions, polynomial type

$$g(t, x, z) = t^2 + 2x^2 - z^2, \tag{2.4}$$

and

$$g(t, x, z) = \arctan \sqrt{\frac{t^2 - x^2 - z^2}{x^2}}, \tag{2.5}$$

as a non-polynomial solution.

For the other example, the Lagrange's system for X_{10} is

$$\frac{dx}{d\varepsilon} = z, \quad \frac{dz}{d\varepsilon} = -x, \quad \frac{dy}{d\varepsilon} = \frac{dt}{d\varepsilon} = \frac{du}{d\varepsilon} = 0. \tag{2.6}$$

Integrating the system (2.6) gives the invariants $y, t, r = x^2 + z^2$ and $u = g(y, t, r)$. Substituting these invariants to Eq. (1.2) gives the following reduced equation

$$-4rg_{rr} + g_{tt} - g_{yy} - 4g_r = 0, \tag{2.7}$$

including two solutions

$$g(r, y, t) = t^2 - y^2 + r, \tag{2.8}$$

and

$$g(r, x, z) = \ln \frac{t - x}{t + x}. \tag{2.9}$$

The proceduer is the same for other symmetries. Results are summarized in Table 1 and 2.

As it is well known that the most important property of symmetries is that they map solutions to solutions. Linear PDEs have an infinite number of solutions and under quite general conditions an admitted symmetry is fiber preserving. To construct the solution one uses the property that the Lie bracket of $X_i, i = 1, ..., 15$ with X_∞ makes another member of the class of symmetries of the form of X_∞. This provide a route to the generation of new and non-trivial solutions from trivial similarity solutions such as are associated with $X_i, i = 1, ..., 11$. It is noteworthy that X_{12}, X_{13}, X_{14} and X_{15} do not give desired solutions. The structure of the new solutions from the property of the Lie bracket with the solution symmetry summarized in Table 3.

For example other solutions could be obtained from the seed solution $\alpha(x, y, z, t) = t^2 + 2X^2 - z^2$ by X_1 that is 0, or by X_2 that are $2z$, 2 and 0, by X_5 that are $2^n ty$, $2^{n-1}(t^2 + y^2)$ and etc. These results also summarized in Table 4, 5, 6 and 7.

3. Non-polynomial Solutions

The seed solutions of X_i is a base for constructing non-polynomial solutions of wave equation. For example for non-polynomial solution (2.9) by X_1 and X_2 is 0 and by X_3 are

$$\alpha^1(x, y, z, t) = \frac{2x}{t^2 - x^2},$$
$$\alpha^2(x, y, z, t) = -\frac{4xt}{(t^2 - x^2)^2},$$
$$\alpha^3(x, y, z, t) = \frac{4x(3t^2 + x^2)}{(t^2 - x^2)^4}.$$

Similarly, X_4 provides the solutions:

$$\alpha^1(x, y, z, t) = \frac{-2t}{t^2 - x^2},$$
$$\alpha^2(x, y, z, t) = \frac{-4xt}{(t - x^2)^2},$$
$$\alpha^3(x, y, z, t) = \frac{-4t(t^2 + 3x^2)}{(t^2 - x^2)^3}.$$

Observation outcomes is expressed in Table 9. We can also run this process for other non-polynpmial solutions in the last column of the Table 2 to obtain a number of solutions for wave equation. These results are coming in Tables 10 to 18.

4. Conclusion

In this paper, by using the Lie symmetry analysis, the symmetry properties and similarity reduction forms of the (3+1)-dimensional linear wave Eq. (1.2) were studied. Moreover, polynomial and non-polynomial solutions of Eq. (1.2) are computed by virtue of this fact, that symmetries and their Lie brackets map solutions to solutions. The method is applicable for any other differential equations which admits symmetries such as X_∞.

TABLE 1. Invariants and the solution set for the symmetry $X_i, i = 1, ..., 15$

Symmetry	Invariant transformations	Reduced equations
X_1	$q = x, p = z, r = t, g = u$	$g_{rr} - g_{qq} - g_{pp} = 0$
X_2	$q = x, p = y, r = t, g = u$	$g_{rr} - g_{qq} - g_{pp} = 0$
X_3	$q = x, p = y, r = z, g = u$	$-g_{qq} - g_{pp} - g_{rr} = 0$
X_4	$q = y, p = z, r = t, g = u$	$g_{rr} - g_{qq} - g_{pp} = 0$
X_5	$q = x, p = z, r = t^2 - y^2, g = u$	$4rg_{rr} + 4g_r - g_{qq} - g_{pp} = 0$
X_6	$q = x, p = y, r = -t^2 + z^2, g = u$	$-4rg_{rr} - 4g_r - g_{qq} - g_{pp} = 0$
X_7	$q = y, p = z, r = t^2 - x^2, g = u$	$4rg_{rr} + 4g_r - g_{qq} - g_{pp} = 0$
X_8	$q = \dfrac{y}{x}, p = \dfrac{z}{x}, r = \dfrac{t}{x}, g = u$	$(q^2+1)g_{qq} + (p^2+1)g_{pp} + (r^2+1)g_{rr}$ $+2g_{pq}pq + 2g_{qr}qr + 2g_{pr}pr + 2g_q q$ $+2g_p p + 2g_r r = 0$
X_9	$q = x, p = t, r = z^2 + y^2, g = u$	$-4g_{rr}r + g_{pp} - g_{qq} - 4g_r = 0$
X_{10}	$q = y, p = t, r = x^2 + z^2, g = u$	$-4g_{rr}r + g_{pp} - g_{qq} - 4g_r = 0$
X_{11}	$q = z, p = t, r = x^2 + y^2, g = u\sqrt{y}$	$-4g_{rr}r - g_{qq} + g_{pp} - 4g_r = 0$
X_{12}	$q = \frac{x}{y}, p = \frac{t}{x}, r = \dfrac{-t^2 + x^2 + y^2 + z^2}{x},$ $g = xu\sqrt{x}$	$\dfrac{-1}{32 - r^3}\left(\dfrac{q^2}{2} + \dfrac{1}{2}\right) g_{qq} + \left(\dfrac{p^2}{2} - \dfrac{1}{2}\right) g_{pp}$ $+g_{qr}qr + g_{pr}pr + \dfrac{g_{rr}r^2}{2} + 2g_q q$ $+g_{pq}pq + 2g_p p + 2g_r r + g = 0$
X_{13}	$q = \frac{z}{x}, p = \frac{t}{x}, r = \dfrac{-t^2 + x^2 + y^2 + z^2}{x},$ $g = xu\sqrt{x}$	$\dfrac{-1}{32 - r^3}\left(\dfrac{q^2}{2} + \dfrac{1}{2}\right) g_{qq} + \left(\dfrac{p^2}{2} - \dfrac{1}{2}\right) g_{pp}$ $+g_{qr}qr + g_{pr}pr + \dfrac{g_{rr}r^2}{2} + 2g_q q$ $+g_{pq}pq + 2g_p p + 2g_r r + g = 0$
X_{14}	$q = \frac{y}{x}, p = \frac{z}{x}, r = \dfrac{t^2 - x^2 - y^2 - z^2}{x},$ $g = xu\sqrt{x}$	$\dfrac{-1}{32 - r^3}\left(\dfrac{q^2}{2} + \dfrac{1}{2}\right) g_{qq} + \left(\dfrac{p^2}{2} - \dfrac{1}{2}\right) g_{pp}$ $+g_{qr}qr + g_{pr}pr + \dfrac{g_{rr}r^2}{2} + 2g_q q$ $+g_{pq}pq + 2g_p p + 2g_r r + g = 0$
X_{15}	$q = \frac{z}{y}, p = \frac{t}{y}, r = \dfrac{-t^2 + x^2 + y^2 + z^2}{y},$ $g = xu\sqrt{x}$	$\dfrac{-1}{32 - r^3}\left(\dfrac{q^2}{2} + \dfrac{1}{2}\right) g_{qq} + \left(\dfrac{p^2}{2} - \dfrac{1}{2}\right) g_{pp}$ $+g_{qr}qr + g_{pr}pr + \dfrac{g_{rr}r^2}{2} + 2g_q q$ $+g_{pq}pq + 2g_p p + 2g_r r + g = 0$

TABLE 2. Invariants and the solution set for the symmetry X_i, $i = 1, ..., 11$

Symmetry	Invariants	Polynomial solutions	Non-polynomial solutions
X_1	t, x, z, u	$t^2 + 2x^2 - z^2$	$\arctan\sqrt{\frac{t^2-x^2-z^2}{x^2}}$
X_2	t, x, y, u	$t^2 + 2x^2 - y^2$	$\arctan\sqrt{\frac{t^2-x^2-y^2}{x^2}}$
X_3	x, y, z, u	$x^2 + y^2 - 2z^2$	$\operatorname{arctanh}\sqrt{\frac{x^2+y^2+z^2}{x^2}}$
X_4	t, y, z, u	$2t^2 + y^2 + z^2$	$\arctan\sqrt{\frac{t^2-y^2-z^2}{z^2}}$
X_5	$x, z, t^2 - y^2, u$	$z^2 + x^2 + t^2 - y^2$	$\arctan\frac{z}{x}$
X_6	$t^2 - z^2, y, x, u$	$t^2 + x^2 + y^2 - z^2$	$\arctan\frac{y}{x}$
X_7	$y, z, t^2 - x^2, u$	$y^2 + t^2 - x^2 + z^2$	$\arctan\frac{z}{y}$
X_8	$\frac{y}{x}, \frac{z}{x}, \frac{t}{x}, u$	1	1
X_9	$x, t, y^2 + z^2, u$	$y^2 + z^2 - x^2 + t^2$	$\ln\frac{t-x}{t+x}$
X_{10}	$y, t, x^2 + z^2, u$	$-y^2 + z^2 + x^2 + t^2$	$\ln\frac{t-y}{t+y}$
X_{11}	$t, z, x^2 + y^2, u$	$y^2 - z^2 + x^2 + t^2$	$\ln\frac{t-z}{t+z}$

TABLE 3. Structure of the new solutions generated by the Lie bracket

$[X_i, X_\infty]$	New Symmetry	New Solutions
$[X_1, X_\infty]$	$\frac{\partial \alpha_{\text{old}}}{\partial y}\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = \frac{\partial \alpha_{\text{old}}}{\partial y}$
$[X_2, X_\infty]$	$\frac{\partial \alpha_{\text{old}}}{\partial z}\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = \frac{\partial \alpha_{\text{old}}}{\partial z}$
$[X_3, X_\infty]$	$\frac{\partial \alpha_{\text{old}}}{\partial t}\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = \frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_4, X_\infty]$	$\frac{\partial \alpha_{\text{old}}}{\partial x}\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = \frac{\partial \alpha_{\text{old}}}{\partial x}$
$[X_5, X_\infty]$	$\left(t\frac{\partial \alpha_{\text{old}}}{\partial y} + y\frac{\partial \alpha_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = t\frac{\partial \alpha_{\text{old}}}{\partial y} + y\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_6, X_\infty]$	$\left(t\frac{\partial \alpha_{\text{old}}}{\partial z} + z\frac{\partial \alpha_{\text{old}}}{\partial t}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = t\frac{\partial \alpha_{\text{old}}}{\partial z} + z\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_7, X_\infty]$	$\left(t\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial t}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = t\frac{\partial \alpha_{\text{old}}}{\partial x} + x\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X8, X_\infty]$	$\Big(x\frac{\partial \alpha_{\text{old}}}{\partial x} + y\frac{\partial \alpha_{\text{old}}}{\partial y} + z\frac{\partial \alpha_{\text{old}}}{\partial z}$ $+t\frac{\partial \alpha_{\text{old}}}{\partial t}\Big)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = x\frac{\partial \alpha_{\text{old}}}{\partial x} + y\frac{\partial \alpha_{\text{old}}}{\partial y} + z\frac{\partial \alpha_{\text{old}}}{\partial z}$ $+t\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_{10}, X_\infty]$	$\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = z\frac{\partial \alpha_{\text{old}}}{\partial y} - y\frac{\partial \alpha_{\text{old}}}{\partial z}$
$[X_{11}, X_\infty]$	$\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = z\frac{\partial \alpha_{\text{old}}}{\partial x} - x\frac{\partial \alpha_{\text{old}}}{\partial z}$
$[X_{12}, X_\infty]$	$\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = y\frac{\partial \alpha_{\text{old}}}{\partial x} - x\frac{\partial \alpha_{\text{old}}}{\partial y}$
$[X_{13}, X_\infty]$	$\Big(zf + xz\frac{\partial f}{\partial x} + yz\frac{\partial f}{\partial y}$ $+\frac{t^2 - x^2 - y^2 + z^2}{2}\frac{\partial f}{\partial z} + tz\frac{\partial f}{\partial t}\Big)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = zf + xz\frac{\partial \alpha_{\text{old}}}{\partial x} + yz\frac{\partial \alpha_{\text{old}}}{\partial y}$ $+\frac{t^2 - x^2 - y^2 + z^2}{2}\frac{\partial \alpha_{\text{old}}}{\partial z} + tz\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_{14}, X_\infty]$	$\Big(yf + xy\frac{\partial f}{\partial x} + yz\frac{\partial f}{\partial z}$ $+\frac{t^2 - x^2 + y^2 - z^2}{2}\frac{\partial f}{\partial y} + ty\frac{\partial f}{\partial t}\Big)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = yf + xy\frac{\partial \alpha_{\text{old}}}{\partial x} + yz\frac{\partial \alpha_{\text{old}}}{\partial z}$ $+\frac{t^2 - x^2 + y^2 - z^2}{2}\frac{\partial \alpha_{\text{old}}}{\partial y} + tz\frac{\partial \alpha_{\text{old}}}{\partial t}$
$[X_{15}, X_\infty]$	$\Big(tf + tx\frac{\partial f}{\partial x} + ty\frac{\partial f}{\partial y}$ $+\frac{t^2 + x^2 + y^2 + z^2}{2}\frac{\partial f}{\partial t} + tz\frac{\partial f}{\partial z}\Big)\frac{\partial}{\partial u}$	$\alpha_{\text{new}} = tf + tx\frac{\partial \alpha_{\text{old}}}{\partial x} + ty\frac{\partial \alpha_{\text{old}}}{\partial y}$ $+\frac{t^2 + x^2 + y^2 + z^2}{2}\frac{\partial \alpha_{\text{old}}}{\partial t} + tz\frac{\partial \alpha_{\text{old}}}{\partial z}$

TABLE 4. Classification of exact polynomial solutions for wave equation

$\alpha_{\text{old}} = t^2 + 2x^2 - z^2$	$\alpha_{\text{old}} = t^2 + 2x^2 - y^2$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}} = 2y$
$\alpha_{\text{new}} = -2z$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = 2t$	$\alpha_{\text{new}} = 2t$
$\alpha_{\text{new}} = 4x$	$\alpha_{\text{new}} = 4x$
$\alpha_{\text{ne}}^n = 2^{2n-1}ty, 2^{2n-1}(t^2 + y^2)$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}}^n = (2)^n tz, 2^{n-1}(t^2 + z^2)$
$\alpha_{\text{new}}^n = 3(2^n)ty, 3(2^n)(t^2 + y^2)$	$\alpha_{\text{new}}^n = 3(2^n)tx, 3(2^n)(t^2 + x^2)$
$\alpha_{\text{new}}^n = 2^n(2x^2 + y^2 - z^2), 4^n(4x^2 + y^2 - z^2)$	$\alpha_{\text{new}}^n = 2^n(t^2 + 2x^2 - y^2), 4^n(t^2 + 2x^2 - y^2)$
$\alpha_{\text{new}}^n = (-1)^{n-1}(2^n)yz, (-1)^n(2^{2n-1})(y^2 - z^2)$	$\alpha_{\text{new}}^n = (-1)^{n-1}(2^n)yz, (-1)^n(2^{2n-1})(y^2 - z^2)$
$\alpha_{\text{new}}^n = (-1)^{n-1}3(2^n)xz, (-1)^{n-1}3(2^n)(x^2 - z^2)$	$\alpha_{\text{new}}^n = (-2)^{i-1}xz, (-2)^{n-1}(x^2 - z^2)$
$\alpha_{\text{new}}^n = (-1)^{(n-1)}4^n xy, (-1)^n 4^n(x^2 - y^2)$	$\alpha_{\text{new}}^n = (-1)^{(n-1)}4^n xz, (-1)^n 4^n(x^2 - z^2)$
$\alpha_{\text{new}} = 2zt^2 + 7x^2z + y^2z - 2z^3$	$\alpha_{\text{new}} = 3(t^2z + 2x^2z - y^2z)$
$\alpha_{\text{new}} = 3t^2y + 6x^2y - 3yz^2$	$\alpha_{\text{new}} = 2t^2y + 7x^2y - 2y^3 + yz^2$
$\alpha_{\text{new}} = 2t^3 + 7tx^2 + ty^2 - 2tz^2$	$\alpha_{\text{new}} = 2t^3 + 7tx^2 - 2ty^2 + tz^2$
$\alpha_{\text{new}} = -(5xt^2 + 4x^3 - 2xy^2 - 5xz^2)$	$\alpha_{\text{new}} = -(5t^2x + 4x^3 - 5xy^2 - 2xz^2)$

Table 5. Classification of exact polynomial solutions for wave equation

$\alpha_{\text{old}} = x^2 + y^2 - 2z^2$	$\alpha_{\text{old}} = 2t^2 + y^2 + z^2$
$\alpha_{\text{new}} = 2y$	$\alpha_{\text{new}} = 2y$
$\alpha_{\text{new}} = -4z$	$\alpha_{\text{new}} = 2z$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}} = 4t$
$\alpha_{\text{new}} = 2x$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}}=(2)^n ty, 2^{n-1}(t^2 + y^2)$	$\alpha^n_{\text{new}} = 3(2^n)yt, 3(2^{n-1})(y^2 + t^2)$
$\alpha^n_{\text{new}}=-(4)^n tz, -(4)^{n-1}(t^2 + z^2)$	$\alpha^n_{\text{new}} =3(2^n)tz, 3(2^{n-1})(t^2 + z^2)$
$\alpha^n_{\text{new}} = (2)^n tx, 2^{n-1}(t^2 + x^2)$	$\alpha^n_{\text{new}} = (4)^n tx, 4^{n-1}(t^2 + x^2)$
$\alpha^n_{\text{new}} = 2^n(x^2 + y^2 - 2z^2)$	$\alpha^n_{\text{new}} = 2^n(2t^2 + y^2 + z^2)$
$\alpha^n_{\text{new}}= (-1)^{n-1}3(2^n)yz, (-1)^{n-1}3(2^{n-1})(y^2 - z^2)$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}}=(-1)^{n-1}3(2^n)xz, (-1)^{n-1}3(2^{n-1})(x^2 - z^2)$	$\alpha^n_{\text{new}} = (-1)^n(2)^n xz, (-1)^n(2)^{n-1}(x^2 - z^2)$
$\alpha_{\text{new}}=0$	$\alpha^n_{\text{new}} = (-1)^n(2)^n xy, (-1)^{n-1}(2)^{n-1}(x^2 - y^2)$
$\alpha_{\text{new}} = -(2zt^2 - 5x^2z - 5y^2z + 4z^3)$	$\alpha_{\text{new}} = 7t^2z - zx^2 + 2zy^2 + 2z^3$
$\alpha_{\text{new}} = t^2y + 2x^2y + 2y^3 - 7yz^2$	$\alpha_{\text{new}} = 7t^2y - x^2y + 2y^3 + 2yz^2$
$\alpha_{\text{new}} = 3tx^2 + 3ty^2 - 6tz^2$	$\alpha_{\text{new}} = 4t^3 + 2tx^2 + 5ty^2 + 5tz^2$
$\alpha_{\text{new}} = -(t^2x + 2x^3 + 2xy^2 - 7xz^2)$	$\alpha_{\text{new}} = -(6t^2x + 3xy^2 + 3xz^2)$

Table 6. Classification of exact polynomial solutions for wave equation

$\alpha_{\text{old}} = z^2 + x^2 + t^2 - y^2$	$\alpha_{\text{old}} = t^2 + x^2 + y^2 - z^2$
$\alpha_{\text{new}}= -2y$	$\alpha_{\text{new}} = 2y$
$\alpha_{\text{new}}=2z$	$\alpha_{\text{new}} = -2z$
$\alpha_{\text{new}}= 2t$	$\alpha_{\text{new}} = 2t$
$\alpha_{\text{new}}= 2x$	$\alpha_{\text{new}} = 2x$
$\alpha_{\text{new}}=0$	$\alpha^n_{\text{new}} = (4^n)ty, (4^{n-1})(t^2 + y^2)$
$\alpha^n_{\text{new}} = (4^n)ty, 4^{n-1}(t^2 + z^2)$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}}=(4^n)tx, (4)^{n-1}(t^2 + x^2)$	$\alpha^n_{\text{new}} = 4^n xt, 4^{n-1}(t^2 + x^2)$
$\alpha^n_{\text{new}} =2^n(t^2 + x^2 - y^2 + 2z^2)$	$\alpha^n_{\text{new}} = 2^n(t^2 + x^2 + 2y^2 - z^2)$
$\alpha^n_{\text{new}}= (-1)^n(4^n)yz, ({}_4)^{n-1}(y^2 - z^2)$	$\alpha^n_{\text{new}} = (-1)^{n-1}(4)^n yz, (-4)^{n-1}(y^2 - z^2)$
$\alpha_{\text{new}}= 0$	$\alpha^n_{\text{new}} = (-1)^{n-1}(4)^n xz, (-4)^{n-1}(x^2 - z^2)$
$\alpha^n_{\text{new}}= (-1)^n(4^n)yx, (4)^{n-1}(x^2 - y^2)$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}}= 4t^2z + 2x^2z - 4y^2z + 2z^3$	$\alpha_{\text{new}} = 2t^2z + 4x^2z + 4y^2z - 2z^3$
$\alpha_{\text{new}}= 2t^2y + 4x^2y - 2y^3 + 4yz^2$	$\alpha_{\text{new}} = 4t^2y + 2x^2y + 2y^3 - 4yz^2$
$\alpha_{\text{new}}= 2t^3 - 4tx^2 - 2ty^2 + 4tz^2$	$\alpha_{\text{new}} = 2t^3 + 4tx^2 - 4ty^2 - 2tz^2$
$\alpha_{\text{new}}= -(4t^2x + 2x^3 - 4xy^2 + 2xz^2)$	$\alpha_{\text{new}} = -(4t^2x + 2x^3 + 2xy^2 - 4xz^2)$

TABLE 7. Classification of exact polynomial solutions for wave equation

$\alpha_{\text{old}} = y^2 + t^2 - x^2 + z^2$	$\alpha_{\text{old}} = y^2 + z^2 + x^2 - t^2$
$\alpha_{\text{new}} = 2y$	$\alpha_{\text{new}} = 2y$
$\alpha_{\text{new}} = 2z$	$\alpha_{\text{new}} = 2z$
$\alpha_{\text{new}} = 2t$	$\alpha_{\text{new}} = -2t$
$\alpha_{\text{new}} = -2x$	$\alpha_{\text{new}} = 2x$
$\alpha^n_{\text{new}} = (4)^n ty, (4^{n-1})(t^2 + y^2)$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}} = (4)^n tz, (4^{n-1})(t^2 + z^2)$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}} = 2^n(t^2 - x^2 + y^2 + 2z^2)$	$\alpha^n_{\text{new}} = 2^n(-t^2 + x^2 + y^2 + z^2)$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}} = (-4)^n xz, (-4)^{n-1}(x^2 - z^2)$	$\alpha_{\text{new}} = 0$
$\alpha^n_{\text{new}} = (-4)^n xy, (-4)^{n-1}(x^2 - y^2$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = 4t^2z - 4x^2z + 2y^2z + 2z^3$	$\alpha_{\text{new}} = -(2t^2z - 2x^2z - 2y^2z - 2z^3$
$\alpha_{\text{new}} = 4t^2y - 4x^2y + 2y^3 + 2yz^2$	$\alpha_{\text{new}} = -(2t^2y - 2x^2y - 2y^3 - 2yz^2)$
$\alpha_{\text{new}} = 2t^3 - 2tx^2 + 4ty^2 + 4tz^2$	$\alpha_{\text{new}} = -(2t^3 - 2tx^2 - 2ty^2 - 2tz^2)$
$\alpha_{\text{new}} = -(2t^2x - 2x^3 + 4xy^2 + 4xz^2)$	$\alpha_{\text{new}} = 2t^2x - 2x^3 - 2xy^2 - 2xz^2$

TABLE 8. Classification of exact polynomial solutions for wave equation

$\alpha_{\text{old}} = -y^2 + z^2 + x^2 + t^2$	$\alpha_{\text{old}} = y^2 - z^2 + x^2 + t^2$
$\alpha_{\text{new}} = -2y$	$\alpha_{\text{new}} = 2y$
$\alpha_{\text{new}} = 2z$	$\alpha_{\text{new}} = -2z$
$\alpha_{\text{new}} = 2t$	$\alpha_{\text{new}} = 2t$
$\alpha_{\text{new}} = 2x$	$\alpha_{\text{new}} = 2x$
$\alpha_{\text{new}} = 0$	$\alpha^n_{\text{new}} = (4)^n ty, (4^{n-1})(t^2 + y^2)$
$\alpha^n_{\text{new}} = (4)^n tz, (4^{n-1})(t^2 + z^2)$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = (4)^n tx, (4^{n-1})(t^2 + x^2)$	$\alpha^n_{\text{new}} = (4)^n tx, (4^{n-1})(t^2 + x^2)$
$\alpha^n_{\text{new}} = 2^n(t^2 + x^2 - y^2 + 2z^2)$	$\alpha^n_{\text{new}} = 2^n(t^2 + x^2 + y^2 - z^2)$
$\alpha^n_{\text{new}} = (-4)^n yz, (-1)^n (4)^{n-1}(y^2 - z^2)$	$\alpha_{\text{new}} = (-1)^{n-1}(4)^n yz, (-1)^{n-1}(-4)^{n-1}(y^2 - z^2)$
$\alpha_{\text{new}} = 0$	$\alpha_{\text{new}} = (-1)^{n-1}(4)^n xz, (-1)^{n-1}(-4)^{n-1}(x^2 - z^2)$
$\alpha^n_{\text{new}} = (-1)^{n-1}(4)^n xy, (-1)^{n-1}(-4)^{n-1}(x^2 - y^2)$	$\alpha_{\text{new}} = 0$
$\alpha_{\text{new}} = 4t^2z + 2x^2z - 4y^2z + 2z^3$	$\alpha_{\text{new}} = 2t^2z - 4x^2z + 4y^2z - 2z^3$
$\alpha_{\text{new}} = 2t^2y + 4x^2y - 2y^3 + 4yz^2$	$\alpha_{\text{new}} = 4t^2y + 2x^2y + 2y^3 - 4yz^2$
$\alpha_{\text{new}} = 2t^3 + 4tx^2 - 2ty^2 + 4tz^2$	$\alpha_{\text{new}} = 2t^3 + 4tx^2 + 4ty^2 - 2tz^2$
$\alpha_{\text{new}} = -(4t^2x + 2x^3 - 4xy^2 + 2xz^2)$	$\alpha_{\text{new}} = -(4t^2x + 2x^3 + 2xy^2 - 4xz^2)$

Table 9. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan\sqrt{\frac{t^2-x^2-z^2}{x^2}}$
X_1	$\alpha_{\text{new}}= 0$
X_2	$\alpha_{\text{new}}= \frac{z}{\sqrt{\frac{t^2-x^2-z^2}{x^2}}(t^2-z^2)} - \frac{(t^4-t^2y^2+t^2z^2-x^2z^2-2z^4)}{(t^2-x^2-z^2)(t^2-y^2)^2\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$
X_3	$\alpha_{\text{new}}= \frac{t}{\sqrt{\frac{t^2-x^2-z^2}{x^2}}(t^2-z^2)^2} - \frac{(2t^4-t^2x^2-t^2z^2-x^2z^2-z^4)}{(t^2-x^2-z^2)(t^2-z^2)^2\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$
X_4	$\alpha_{\text{new}}= \frac{1}{x\sqrt{\frac{t^2-x^2-z^2}{x^2}}}, \frac{1}{t^2-x^2-z^2)x\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc
X_5	$\alpha_{\text{new}}= \frac{t}{\sqrt{\frac{t^2-x^2-z^2}{x^2}}(t^2-z^2)^2}$
X_6	$\alpha_{\text{new}}= 0$
X_7	$\alpha_{\text{new}}= -\frac{t(t^2-y^2-z^2)}{x(t^2-z^2)\sqrt{\frac{t^2-x^2-z^2}{x^2}}}, \frac{(t^4-t^2x^2-x^2z^2-z^4)}{t^2-z^2)^2\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc
X_8	$\alpha_{\text{new}}= 0$
X_9	$\alpha_{\text{new}}=\frac{yz}{(t^2-z^2)\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$, etc
X_{10}	$\alpha_{\text{new}}= \frac{z(t^2-x^2-z^2)}{x(t^2-z^2)\sqrt{\frac{t^2-x^2-z^2}{x^2}}}, \frac{(t^4-t^2x^2-x^2z^2-z^4)}{t^2-z^2)\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc
X_{11}	$\alpha_{\text{new}}=\frac{y}{x\sqrt{\frac{t^2-x^2-z^2}{x^2}}}, \frac{(t^2-x^2-y^2-z^2)}{(t^2-x^2-z^2)\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc

TABLE 10. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}$
X_1	$\alpha_{\text{new}} = -\dfrac{y}{\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}(t^2-y^2)}$ $-\dfrac{(t^4-t^2x^2+t^2y^2-x^2y^2-2z^4)}{(t^2-x^2-y^2)(t^2-y^2)^2\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$,etc
X_2	$\alpha_{\text{new}} = 0$
X_3	$\alpha_{\text{new}} = \dfrac{t}{\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}(t^2-y^2)}$ $-\dfrac{(2t^4-t^2x^2-t^2y^2-y^4)}{(t^2-x^2-y^2)(t^2-y^2)^2\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$
X_4	$\alpha_{\text{new}} = -\dfrac{1}{x\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, $\dfrac{1}{(t^2-x^2-y^2)x\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$,etc
X_5	$\alpha_{\text{new}} = 0$
X_6	$\alpha_{\text{new}} = \dfrac{zt}{(t^2-y^2)\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, etc
X_7	$\alpha_{\text{new}} = -\dfrac{t(t^2-y^2-x^2)}{x(t^2-y^2)\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, $\dfrac{(t^4-t^2x^2-x^2y^2-y^4)}{t^2-y^2)^2\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$,etc
X_8	$\alpha_{\text{new}} = 0$
X_9	$\alpha_{\text{new}} = \dfrac{yz}{(t^2-y^2)\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, etc
X_{10}	$\alpha_{\text{new}} = -\dfrac{y(t^2-x^2-y^2)}{x(t^2-y^2)\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, $\dfrac{(t^4-t^2x^2-x^2z^2-z^4)}{t^2-y^2)^2\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$,etc
X_{11}	$\alpha_{\text{new}} = \dfrac{y}{x\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$, $\dfrac{(t^2-x^2-y^2-z^2)}{(t^2-x^2-y^2)\sqrt{\dfrac{t^2-x^2-y^2}{x^2}}}$,etc

TABLE 11. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \operatorname{arctanh}\sqrt{\frac{x^2+y^2+z^2}{x^2}}$
X_1	$\alpha_{\text{new}}= \frac{y}{\sqrt{\frac{x^2+y^2+z^2}{x^2}}(y^2+z^2)}$ $\frac{(x^2y^2-x^2z^2+2y^4+y^2z^2-z^4)}{(x^2+y^2+z^2)(y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{x^2}}}$
X_2	$\alpha_{\text{new}}= \frac{z}{\sqrt{\frac{x^2+y^2+z^2}{x^2}}(y^2+z^2)}$ $\frac{(x^2y^2-x^2z^2+y^4-y^2z^2-2z^4)}{(x^2+y^2+z^2)(y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{x^2}}}$
X_3	$\alpha_{\text{new}}= 0$
X_4	$\alpha_{\text{new}}= \frac{1}{x\sqrt{\frac{x^2+y^2+z^2}{x^2}}},$ $\frac{1}{x^2+y^2+z^2)x\sqrt{\frac{x^2+y^2+z^2}{x^2}}}$,etc
X_5	$\alpha_{\text{new}}=\frac{2y}{(y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{x^2}}}$, etc
X_6	$\alpha_{\text{new}}=\frac{tz}{(y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{x^2}}},$
X_7	$\alpha_{\text{new}}=\frac{t}{(x)\sqrt{\frac{x^2+y^2+z^2}{x^2}}},$ $\alpha_{\text{new}}=-\frac{t^2-x^2-y^2-z^2}{(x^2+y^2+z^2)\sqrt{\frac{x^2+y^2+z^2}{x^2}}}$,etc
X_8	$\alpha_{\text{new}}=0$
X_9	$\alpha_{\text{new}}=0$
X_{10}	$\alpha_{\text{new}}=\frac{z(x^2+y^2+z^2)}{x(y^2+z^2)\sqrt{\frac{t^2+y^2+z^2}{x^2}}},$ $-\frac{(x^2y^2-x^2z^2+y^4-z^4)}{(y^2+z^2)^2\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc
X_{11}	$\alpha_{\text{new}}=\frac{y(x^2+y^2+z^2)}{x(y^2+z^2)\sqrt{\frac{t^2+y^2+z^2}{x^2}}},$ $\frac{(x^2y^2-x^2z^2+y^4-z^4)}{(y^2+z^2)^2\sqrt{\frac{t^2-x^2-z^2}{x^2}}}$,etc

TABLE 12. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}$
X_1	$\alpha_{\text{new}}= \dfrac{y}{\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}(t^2-y^2)} - \dfrac{(t^4+t^2y^2-t^2z^2-2y^4-y^2z^2)}{(x^2+y^2+z^2)(t^2-y^2)^2\sqrt{\dfrac{x^2+y^2+z^2}{x^2}}}$
X_2	$\alpha_{\text{new}}= \dfrac{1}{z\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}, \dfrac{1}{(t^2-y^2-z^2)x\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$,etc
X_3	$\alpha_{\text{new}}= \dfrac{t}{\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}(t^2-y^2)} - \dfrac{(2t^4-t^2y^2-t^2z^2-y^4-y^2z^2)}{(x^2+y^2+z^2)(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$
X_4	$\alpha_{\text{new}}= 0$
X_5	$\alpha_{\text{new}}= 0$
X_6	$\alpha_{\text{new}}= -\dfrac{t(t^2-y^2-z^2)}{z(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}, \dfrac{(t^4-t^2z^2-y^4-y^2z^2)}{(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$,etc
X_7	$\alpha_{\text{new}}=\dfrac{xt}{(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$, etc
X_8	$\alpha_{\text{new}}=0$
X_9	$\alpha_{\text{new}}= \dfrac{y(t^2-y^2-z^2)}{z(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}, \dfrac{(t^4-t^2z^2-y^4-y^2z^2)}{(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$,etc
X_{10}	$\alpha_{\text{new}}=\dfrac{x}{z\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}, \dfrac{(t^2-x^2-y^2-z^2)}{(t^2-y^2-z^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$,etc
X_{11}	$\alpha_{\text{new}}=\dfrac{xy}{(t^2-y^2)\sqrt{\dfrac{t^2-y^2-z^2}{z^2}}}$, etc

TABLE 13. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan\dfrac{z}{x}$
X_1	$\alpha_{\text{new}} = 0$
X_2	$\alpha^n_{\text{new}} = \dfrac{(n-1)!x(x^2-3z^2)^{n-2}}{(x^2+z^2)^n}, \frac{n!xz(x^2-z^2)^{\frac{n}{2}-1}}{(x^2+z^2)^n}$
X_3	$\alpha_{\text{new}} = 0$
X_4	$\alpha^n_{\text{new}} = -\dfrac{(n-1)!z(3x^2-z^2)^{n-2}}{(x^2+z^2)^n}, \frac{n!xz(x^2-z^2)^{\frac{n}{2}-1}}{(x^2+z^2)^n}$
X_5	$\alpha_{\text{new}} = 0$
X_6	$\alpha_{\text{new}} = \dfrac{tx}{x^2+z^2}, -\dfrac{xz(2t^2-x^2-z^2)}{(x^2+z^2)^2}$,etc
X_7	$\alpha_{\text{new}} = \dfrac{-tz}{x^2+z^2}, \dfrac{xz(2t^2-x^2-z^2)}{(x^2+z^2)^2}$,etc
X_8	$\alpha_{\text{new}} = 0$
X_9	$\alpha_{\text{new}} = \dfrac{-yx}{x^2+z^2}, -\dfrac{xz(2y^2+x^2+z^2)}{(x^2+z^2)^2}$,etc
X_{10}	$\alpha_{\text{new}} = -1, 0$
X_{11}	$\alpha_{\text{new}} = \dfrac{-yz}{x^2+z^2}, \dfrac{xz(2y^2+x^2+z^2)}{(x^2+z^2)^2}$,etc

TABLE 14. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan \frac{y}{x}$
X_1	$\alpha_{\text{new}}^n = \frac{(n-1)!x(x^2-3y^2)^{n-2}}{(x^2+y^2)^n}$, $-\frac{n!xy(x^2-y^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n}$
X_2	$\alpha_{\text{new}} = 0$
X_3	$\alpha_{\text{new}} = 0$
X_4	$\alpha_{\text{new}}^n = -\frac{(n-1)!y(3x^2-y^2)^{n-2}}{(x^2+y^2)^n}$, $\frac{n!xz(x^2-y^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n}$
X_5	$\alpha_{\text{new}} = \frac{tx}{x^2+y^2}, -\frac{xy(2t^2-x^2-y^2)}{(x^2+y^2)^2}$,etc
X_6	$\alpha_{\text{new}} = 0$
X_7	$\alpha_{\text{new}} = \frac{-ty}{x^2+y^2}, \frac{xy(2t^2-x^2-y^2)}{(x^2+y^2)^2}$,etc
X_8	$\alpha_{\text{new}} = 0$
X_9	$\alpha_{\text{new}} = \frac{-yx}{x^2+y^2}, -\frac{xy(2y^2+x^2+z^2)}{(x^2+y^2)^2}$,etc
X_{10}	$\alpha_{\text{new}} = 0$
X_{11}	$\alpha_{\text{new}} = \frac{-yz}{x^2+y^2}, \frac{xy(y^2+x^2+2z^2)}{(x^2+y^2)^2}$,etc

TABLE 15. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \arctan\frac{z}{y}$
X_1	$\alpha^n_{\text{new}}=\dfrac{(n-1)!z(3y^2-z^2)^{n-2}}{(y^2+z^2)^n}, -\frac{n!yz(y^2-z^2)^{\frac{n}{2}-1}}{(y^2+z^2)^n}$
X_2	$\alpha^n_{\text{new}}=-\dfrac{(n-1)!y(y^2-3z^2)^{n-2}}{(y^2+z^2)^n}, \frac{n!xy(x^2-z^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n}$
X_3	$\alpha_{\text{new}}=0$
X_4	$\alpha_{\text{new}}=0$
X_5	$\alpha_{\text{new}}=\dfrac{-tz}{x^2+y^2}, -\dfrac{yz(2t^2-y^2-z^2)}{(x^2+y^2)^2}$,etc
X_6	$\alpha_{\text{new}}=\dfrac{ty}{x^2+y^2}, -\dfrac{yz(2t^2-x^2-z^2)}{(x^2+y^2)^2}$,etc
X_7	$\alpha_{\text{new}}=0$
X_8	$\alpha_{\text{new}}=0$
X_9	$\alpha_{\text{new}}=-1,0$
X_{10}	$\alpha_{\text{new}}=\dfrac{-yx}{x^2+y^2}, \dfrac{yz(2x^2+y^2+z^2)}{(x^2+y^2)^2}$,etc
X_{11}	$\alpha_{\text{new}}=\dfrac{yz}{x^2+y^2}, \dfrac{zy(2x^2+z^2+y^2)}{(x^2+y^2)^2}$,etc

TABLE 16. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \ln \frac{t-x}{t+x}$
X_1	$\alpha_{\text{new}} = 0$
X_2	$\alpha_{\text{new}} = 0$
X_3	$\alpha^n_{\text{new}} = \frac{2(2)^{n-1}x(3t^2+x^2)^{n-2}}{(t^2-x^2)^n}, \frac{2n!tx(t^2+x^2)^{\frac{n}{2}-1}}{(t^2-x^2)^n}$
X_4	$\alpha^n_{\text{new}} = -\frac{2(2)^{n-1}t(3x^2+t^2)^{n-2}}{(t^2-x^2)^n}, \frac{2n!xt(x^2+t^2)^{\frac{n}{2}-1}}{(t^2-x^2)^n}$
X_5	$\alpha_{\text{new}} = \frac{2xy}{t^2-x^2}, \frac{2tx(t^2-x^2-2y^2)}{(t^2-x^2)^2}$,etc
X_6	$\alpha_{\text{new}} = \frac{2xz}{t^2-x^2}, \frac{2tx(t^2-x^2-2z^2)}{(t^2-x^2)^2}$,etc
X_7	$\alpha_{\text{new}} = -2, 0$
X_8	$\alpha_{\text{new}} = 0$
X_9	$\alpha_{\text{new}} = 0$
X_{10}	$\alpha_{\text{new}} = \frac{-2tz}{t^2-x^2}, \frac{2tx(t^2-x^2-2z^2)}{(t^2-x^2)^2}$,etc
X_{11}	$\alpha_{\text{new}} = \frac{-2ty}{t^2-x^2}, \frac{2tx(t^2-x^2+2y^2)}{(t^2-x^2)^2}$,etc

TABLE 17. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \ln \frac{t-y}{t+y}$
X_1	$\alpha^n_{\text{new}}=\dfrac{2(2)^{n-1}t(3t^2+x^2)^{n-2}}{(t^2-y^2)^n}$, $\frac{2n!ty(t^2+x^2)^{\frac{n}{2}-1}}{(t^2-y^2)^n}$
X_2	$\alpha_{\text{new}}= 0$
X_3	$\alpha^n_{\text{new}}=\dfrac{2(2)^{n-1}x(t^2+3x^2)^{n-2}}{(t^2-y^2)^n}$, $\frac{2n!ty(t^2+x^2)^{\frac{n}{2}-1}}{(t^2-y^2)^n}$
X_4	$\alpha_{\text{new}}=0$
X_5	$\alpha_{\text{new}}=0$
X_6	$\alpha_{\text{new}}=\dfrac{2yz}{t^2-y^2}, \dfrac{2ty(t^2-y^2-2z^2)}{(t^2-y^2)^2}$,etc
X_7	$\alpha_{\text{new}}=\dfrac{2xy}{t^2-y^2}, \dfrac{2ty(t^2-2x^2-y^2)}{(t^2-y^2)^2}$,etc
X_8	$\alpha_{\text{new}}=0$
X_9	$\alpha_{\text{new}}=\dfrac{-2tz}{t^2-y^2}, \dfrac{2ty(t^2-y^2-2z^2)}{(t^2-y^2)^2}$,etc
X_{10}	$\alpha_{\text{new}}=0$
X_{11}	$\alpha_{\text{new}}=\dfrac{2tx}{t^2-y^2}, \dfrac{2ty(t^2-2x^2-y^2)}{(t^2-y^2)^2}$,etc

TABLE 18. Classification of exact non-polynomial solutions for wave equation

X_i	$\alpha_{\text{old}} = \ln \frac{t-z}{t+z}$
X_1	$\alpha_{\text{new}} = 0$
X_2	$\alpha^n_{\text{new}} = -\dfrac{2(2)^{n-1}t(t^2+3z^2)^{n-2}}{(t^2-z^2)^n}, -\frac{2n!tz(t^2+z^2)^{\frac{n}{2}-1}}{(t^2-z^2)^n}$
X_3	$\alpha^n_{\text{new}} = -\dfrac{2(2)^{n-1}z(t^2+3x^2)^{n-2}}{(t^2-z^2)^n}, \frac{2n!tz(t^2+x^2)^{\frac{n}{2}-1}}{(t^2-z^2)^n}$
X_4	$\alpha_{\text{new}} = 0$
X_5	$\alpha_{\text{new}} = \dfrac{2yz}{t^2-x^2}, \dfrac{2tz(t^2-2y^2-z^2)}{(t^2-z^2)^2}$,etc
X_6	$\alpha_{\text{new}} = -2, 0$
X_7	$\alpha_{\text{new}} = \dfrac{2xz}{t^2-z^2}, \dfrac{2tz(t^2-2x^2-z^2)}{(t^2-z^2)^2}$,etc
X_8	$\alpha_{\text{new}} = 0$
X_9	$\alpha_{\text{new}} = \dfrac{-2ty}{t^2-z^2}, \dfrac{2tz(t^2-2y^2-z^2)}{(t^2-z^2)^2}$,etc
X_{10}	$\alpha_{\text{new}} = \dfrac{2tx}{t^2-z^2}, \dfrac{2tz(t^2-2x^2-z^2)}{(t^2-z^2)^2}$,etc
X_{11}	$\alpha_{\text{new}} = 0$

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