Journal of Advanced Mathematical Modeling
ISSN (print): 2251-8088, ISSN (on-line): 2645-6141
Vol. 13 No. 4 (2024), pp. 23-28.
DOI: 10.22055/JAMM.2023.43461.2150

AN EIGENVALUE OPTIMIZATION PROBLEM FOR DIRICHLET-LAPLACIAN WITH A DRIFT

MOHSEN ZIVARI-REZAPOUR

Communicated by: F. Azarpanah

Abstract. In this note, we prove a monotonicity result related to the principal eigenvalue for Dirichlet-Laplacian with a drift operator in a punctured ball.

1. Introduction

Let B be the unit ball in $\mathbb{R}^n$ centered at the origin, B_h is the ball centered at $(h, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with radius $r < 1$ and $D_h := B \setminus \overline{B_h}$, $0 \leq h \leq 1 - r$. Here $\overline{E}$ denotes the clouser of the set $E \subset \mathbb{R}^n$. Let $L := \Delta + x \cdot \nabla$ be the Laplace operator with a drift. We consider the following eigenvalue problem

$$\begin{cases} -Lu(x) = \lambda u(x) & \text{for } x \in D_h, \\ u(x) = 0 & \text{for } x \in \partial D_h, \end{cases} \tag{1.1}$$

or equivalently in the weighted eigenvalue problem

$$\begin{cases} -\nabla \cdot \left(e^{|x|^2/2} \nabla u(x) \right) = \lambda e^{|x|^2/2} u(x) & \text{for } x \in D_h, \\ u(x) = 0 & \text{for } x \in \partial D_h. \end{cases} \tag{1.2}$$

We say $(\lambda, u) \in \mathbb{R} \times H_0^1(D_h)$ is an eigenpair for the problem (1.1) whenever

$$\int_{D_h} e^{|x|^2/2} \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_{D_h} e^{|x|^2/2} uv \, \mathrm{d}x, \tag{1.3}$$

MSC(2020): 35J20, 49R05.
Keywords: Monotonicity, Eigenvalue, Shape derivative, Laplacian.
Received: 11 April 2023, Accepted: 18 December 2023.

for all $v \in H_0^1(D_h)$. The real number λ is eigenvalue and the function u is called an eigenfunction associated to it.

One of the physical phenomena of the problem (1.1) is the vibration of an elastic membrane. Assume that D_h is a planar region occupied by an elastic membrane fixed around the boundary. It is well-known that natural frequencies of the membrane are related to the eigenvalues λ.

Let

$$W_h := \left\{ w \in H_0^1(D_h) : \int_{D_h} e^{|x|^2/2} w^2(x) \, \mathrm{d}x = 1 \right\}.$$

The principal (first) eigenvalue of (1.1) is

$$\lambda_1(h) = \inf \left\{ \int_{D_h} e^{|x|^2/2} |\nabla w(x)|^2 \, \mathrm{d}x : w \in W_h \right\}.$$

It is weel-known that the minimum is attained. Let $u_h \in W_h$ be a minimizer. Since $|u_h|$ is a minimizer as well, we can assume that u_h is non-negative in D_h. By a strong maximum principle we find that u_h is positive in D_h. Also, It is unique up to a constant factor of itself (see [6, 1, 9] for more details). In this note, we show that $\lambda_1(h)$ is decreasing in $0 \leq h \leq 1 - r$. To do this end, we use the shape derivative, see [11, 12, 3, 2, 4, 8], and deduce that

$$\dot{\lambda_1}(h) = -\int_{\partial D_h} e^{|x|^2/2} (\frac{\partial u_h}{\partial \nu})^2 \nu_1 \, \mathrm{d}S,$$

where $\dot{\lambda_1}(h)$ denotes the derivative of λ_1 with respect to h, ν stands for unit outward normal on ∂D_h and ν_1 is the first component of ν. Then by the Walter's maximum principle [13, Theorem 2] we prove that $\dot{\lambda_1}(h) \leq 0$ for $0 \leq h \leq 1 - r$. Note that we unable to show the strict monotonicity.

If $L = \Delta$ in (1.1), this result has been proved by Ashbaugh and Chatelain (personal communication, 2012), Harrell *et al.* [5], Kesavan [7], and Ramm and Shivakumar [10]. In [3], we tried to extend the result of Ramm and Shivakumar [10] for Dirichlet p-Laplacian eigenvalue problem but we were able to do that for weighted eigenvalue problem with a sign changing weight. Finally, Chorwadwala and Mahadevan [2] extend it. However, they were also unable to conclude the strict monotonicity.

2. Main result

The main result of the paper is the following

Theorem 2.1. *Assume $(\lambda_1(h), u_h) \in \mathbb{R}^+ \times W_h$ be the principal eigenpair of the eigenvalue problem (1.1). If $u_h \in C^2(D_h) \cap C(\overline{D_h})$ then $\lambda_1(h)$ is decreasing for $0 \le h \le 1-r$.*

Proof. We use the shape derivative. Let $\dot{u}_h := \frac{du_h}{dh}$ and $\dot{\lambda_1}(h) := \frac{d\lambda_1}{dh}$. We have

$$\begin{cases} -\nabla \cdot (e^{|x|^2/2}\nabla \dot{u}_h(x)) = e^{|x|^2/2}(\dot{\lambda_1}(h)u_h(x) + \lambda_1(h)\dot{u}_h(x)) & \text{for } x \in D_h, \\ \dot{u}_h(x) = -\frac{\partial u_h}{\partial \nu}(x)\nu_1(x) & \text{for } x \in \partial D_h, \end{cases} \tag{2.1}$$

where the boundary equation in (2.1) follows directly from [11, Theorem 3.2].

By multiply the differential equations in (1.2) and (2.1) by $\dot{u}_h$ and u_h respectively, and integrate the results over D_h we infer that

$$\int_{D_h} e^{|x|^2/2}\nabla u_h \cdot \nabla \dot{u}_h \, \mathrm{d}x + \int_{\partial D_h} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu})^2 \nu_1 \, \mathrm{d}S = \lambda_1(h) \int_{D_h} e^{|x|^2/2} u_h \dot{u}_h \, \mathrm{d}x, \tag{2.2}$$

and

$$\int_{D_h} e^{|x|^2/2}\nabla u_h \cdot \nabla \dot{u}_h \, \mathrm{d}x = \lambda_1(h) \int_{D_h} e^{|x|^2/2} u_h \dot{u}_h \, \mathrm{d}x + \dot{\lambda_1}(h) \int_{D_h} e^{|x|^2/2} u_h^2 \, \mathrm{d}x. \tag{2.3}$$

Since $u_h \in W_h$ by subtracting (2.2) and (2.3) we deduce

$$\dot{\lambda_1}(h) = -\int_{\partial D_h} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu})^2 \nu_1 \, \mathrm{d}S. \tag{2.4}$$

By symmetry we have

$$\int_{\partial B^+} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu})^2 \nu_1(x) \, \mathrm{d}S = -\int_{\partial B^-} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu})^2 \nu_1(x) \, \mathrm{d}S,$$

where $\partial B^+ := \{x \in \partial B : x_1 \ge 0\}$ and $\partial B^- := \{x \in \partial B : x_1 \le 0\}$. Therefore

$$\dot{\lambda_1}(h) = -\int_{\partial B_h} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu})^2 \nu_1 \, \mathrm{d}S.$$

It's clear by symmetry that $\dot{\lambda_1}(0) = 0$. We show that $\dot{\lambda_1}(h) \le 0$ for $0 \le h \le 1-r$. To do this, we shift the coordinate axes so that the hyperplane $l = \{x \in \mathbb{R}^n : x_1 = 0\}$ passes through the center of B_h. Now, let $A := \{x \in D_h : x_1 > 0\}$ and A^* be the image of it with respect to l. Also, let $\partial B_h^+ := \{x \in \partial B_h : x_1 \ge 0\}$ and $\partial B_h^- := \{x \in \partial B_h : x_1 \le 0\}$. We define the function

$$v(x) = \begin{cases} u_h(x) & \text{if } x \in \overline{A}, \\ u_h(x_l) & \text{if } x \in \overline{A^*}, \end{cases} \tag{2.5}$$

where x_l is the reflection of x with respect to l. For $x \in A^*$, since $|x| = |x_l|$ we have

$$\begin{aligned} -\nabla \cdot (e^{|x|^2/2}\nabla v(x)) &= -\nabla \cdot (e^{|x_l|^2/2}\nabla u_h(x_l)) \\ &= \lambda_1(h)u_h(x_l)e^{|x_l|^2/2} \\ &= \lambda_1(h)v(x)e^{|x|^2/2}. \end{aligned} \tag{2.6}$$

Let $w := u_h - v$. From (2.6) and (1.2) we have

$$Lw + \lambda_1(h)w = 0 \text{ in } A^*, w \geq 0 \text{ on } \partial A^*, \tag{2.7}$$

and

$$Lv + \lambda_1(h)v = 0 \text{ and } v > 0 \text{ in } A^*. \tag{2.8}$$

Therefore, since $u_h \in C^2(D_h) \cap C(\overline{D_h})$, by the Walter's maximum principle [13, Theorem 2] we deduce either $w = \beta v$ in A^* for some $\beta < 0$ or $w \geq 0$ in A^*. Assume $w = \beta v$ in A^*. Let $x \in \partial A \cap \partial B^+$ be fixed. So there exists a sequence $\{x^{(k)}\} \subset A$ such that $|x^{(k)} - x| \to 0$ as $k \to \infty$. From $u_h \in C(\overline{D_h})$, we infer that

$$0 < u_h(x_l) = \lim_{k\to\infty} u_h(x_l^{(k)}) = \lim_{k\to\infty} \Big(w(x_l^{(k)}) + v(x_l^{(k)})\Big) = (\beta + 1)v(x_l) = 0.$$

This is a contradiction. Hence $w \geq 0$ in A^*. Thus $u_h \geq v$ in A^*. Since $u_h = v = 0$ on ∂B_h^-, for $z \in \partial B_h^-$ we obtain

$$\frac{\partial}{\partial \nu}(u_h - v)(z) = \lim_{t\to 0^-} \frac{(u_h - v)(z + t\nu) - (u_h - v)(z)}{t} \leq 0.$$

Thus $\frac{\partial u_h}{\partial \nu} \leq \frac{\partial v}{\partial \nu}$ on ∂B_h^-. Since $\frac{\partial v}{\partial \nu} \leq 0$ on ∂B_h^-, we infer that

$$\left|\frac{\partial u_h}{\partial \nu}\right| \geq \left|\frac{\partial v}{\partial \nu}\right| \text{ on } \partial B_h^-. \tag{2.9}$$

Now from (2.9) we infer that

$$\begin{aligned} \int_{\partial B_h^-} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu}(x))^2 \nu_1(x) \, \mathrm{d}S &\geq \int_{\partial B_h^-} e^{|x|^2/2}(\frac{\partial v}{\partial \nu}(x))^2 \nu_1(x) \, \mathrm{d}S \\ &= \int_{\partial B_h^-} e^{|x_l|^2/2}(\frac{\partial u_h}{\partial \nu}(x_l))^2 \nu_1(x) \, \mathrm{d}S \\ &= \int_{\partial B_h^+} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu}(x))^2 \nu_1(x_l) \, \mathrm{d}S \\ &= -\int_{\partial B_h^+} e^{|x|^2/2}(\frac{\partial u_h}{\partial \nu}(x))^2 \nu_1(x) \, \mathrm{d}S. \end{aligned}$$

Therefore $\dot{\lambda}_1(h) \leq 0$. □

Conclusion

According to the numerical results in [10], it seems that, also in our problem, λ_1 be strictly decreasing, but for proving we will need probably a strong comparison principle which conclude that

$$\left|\frac{\partial u_h}{\partial \nu}\right| > \left|\frac{\partial v}{\partial \nu}\right|,$$

in a neighbourhood of $\partial B_h^- \cap A^*$.

Acknowledgments

I am grateful to the Research Council of Shahid Chamran University of Ahvaz for the research grant SCU.MM1402.441.

References

[1] Bozorgnia, F., Mohammadi, S. A. and Vejchodský, T., 2019. The first eigenvalue and eigenfunction of a nonlinear elliptic system. *Applied Numerical Mathematics, 145*, pp.159-174. doi:10.1016/j.apnum.2019.06.004

[2] Chorwadwala, A.M.H. and Mahadevan R., 2015. An eigenvalue optimization problem for the p-Laplacian. *Proc. R. Soc. Edinb. A, 145*, pp.1145-1151. doi: 10.1017/S0308210515000232

[3] Emamizadeh, B. and Zivari-Rezapour, M., 2007. Monotonicity of the principal eigenvalue of the p-Laplacian in an annulus. *Proc. Am. Math. Soc., 136*,pp.1325-1331. doi:10.1090/S0002-9939-07-09153-8

[4] García-Melián, J. and Sabina de Lis, J., 2001. On the perturbation of eigenvalues for the p-Laplacian. *C. R. Acad. Sci. Paris Sér. I, 332*, pp.893-898. doi:10.1016/S0764-4442(01)01956-5

[5] Harrell, E. M., Kröger, P. and Kurata, K., 2001. On the placement of an obstacle or a well as to optimize the fundamental eigenvalue. *SIAM J. Math. Analysis, 33* , pp.240-259. doi:10.1137/S00361410993575

[6] Henrot, A., 2006. *Extremum problems for eigenvalues of elliptic operators*, Springer. doi:10.1007/3-7643-7706-2

[7] Kesavan S., 2003. On two functionals connected to the Laplacian in a class of doubly connected domains. *Proc. R. Soc. Edinb. A, 133*, pp.617-624. doi:10.1017/S0308210500002560

[8] Lamberti, P. D., 2003. A differentiability result for the first eigenvalue of the p-Laplacian upon domain perturbation. *Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday, vol. I, II*, pp.741-754.

[9] Mohammadi, S. A., Bozorgnia, F. and Voss, H., 2019. Optimal Shape Design for the p-Laplacian Eigenvalue Problem. *Journal of Scientific Computing, 78*, pp.1231-1249. doi:10.1007/s10915-018-0806-7

[10] Ramm, A. G. and Shivakumar, P. N., 1998. Inequalities for the minimal eigenvalue of the Laplacian in an annulus. *Math. Inequalities Applic., 1*, pp.559-563. doi:10.48550/arXiv.math-ph/9911040

[11] Simon, J., 1980. Differentiation with respect to the domain in boundary value problems. *Numer. Funct. Anal. and Optimiz., 2*, pp.649-687. doi:10.1080/01630563.1980.10120631

[12] Sokolowski, J. and Zolesio, J. P., 1992. *Introduction to shape optimization: shape sensitivity analysis*. Springer Series in Computational Mathematics, vol. 10. doi:10.1007/978-3-642-58106-9

[13] Walter, W., 1989. A theorem on elliptic differential inequalities with an application to gradient bounds. *Math. Z., 200*, pp.293-299. doi:10.1007/BF01230289

Mohsen Zivari-Rezapour
Department of Mathematics,
Faculty of Mathematical Sciences and Computer,
Shahid Chamran University of Ahvaz,
Ahvaz, Iran
Email: mzivari@scu.ac.ir