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KARAMZADEH'S CAPTIVATING THOUGHTS AND HIS ELEGANT SOLUTIONS TO SOME POPULAR AND CLASSICAL RESULTS REVISITED: SOME COMMENTS AND EXPLANATIONS

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Dedicated to Professor O.A.S. Karamzadeh on the occasion of his 80th birthday

Communicated by: A. Azarang

Abstract. In this note, we intend to explain and interpret the proofs of some of the classic mathematical results that are made easy by the beautiful mind of Karamzadeh, who has an incredible love for mathematics and has been enthusiastically busy for years with regard to its popularization. Concerning the latter comment, the general readers are the primary focus of our attention.

1. **Introduction**

Before dealing with the title, let us cite the last comment in [37], which is related to Karamzadeh's thoughts, too "Let me conclude my letter with related advice my grandfather gave my mom (he also happens to be a mathematics professor): If you intend to become a mathematician, try in general to invent new tools for resolving problems that exist naturally, instead of creating artificial problems (even suitable ones) that can be resolved by the tools that you already have". Incidentally, the author of [37] is the oldest grandson of Karamzadeh whom I met recently at AIMC 54 (note, AIMC stands briefly for the annual Iranian mathematics conference), in Zanjan, where he admitted enthusiastically that he wrote that note with the help of his mom and his grandfather. As a former

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student, a colleague, and a collaborator of Karamzadeh, I cannot help claiming that in all these years, Karamzadeh has always been heeding his own advice, too. In particular, he also believes that every mathematician should be able and interested to explain and communicate his or her ideas in a way which is comprehensible to others (i.e. with enough explanations), especially when they (i.e., the audiences and the listeners) are with less knowledge and might not have the necessary background for the subject of discussions and therefore they may not easily comprehend the discussion. Indeed, he believes that mathematics is well-defined doodling, and this doodling is never complete unless it can be shared by laymen, see [17, footnote]. He used to emphasize during our private talks, see the acknowledgments in [29], that of all the published results in the current trends in mathematics, in our expertise in the literature, only those remain useful and might appear in the future textbooks which are breakthrough discovery, or their proofs use some unprecedented elements, or they introduced a fundamental concept which is to be of use for many years to come. Otherwise, in the next century most of our published results will be treated as nowadays we deal with the articles concerning the tables which contain the four-place common logarithms and antilogarithms. However, Pythagoras Theorem will remain interesting for ever in the literature. As for other elementary results similar to the latter theorem, those which receive different proofs, every now and then, by some well-known mathematicians (due to their importance and their applications) seem to be more interesting and basic enough to remain for a long time in the literature and, in particular, they are more likely to appear in the high-school textbooks. Karamzadeh parallel to doing research in algebra, topology, and mathematics education, has always been interested in dealing with latter kind of elementary results and has already provided elegant proofs to some of them, which I believe will remain in the history of mathematics for a long time. He usually says he may share the feeling of euphoria concerning some of his results in algebra and topology with only a limited number of experts in these areas, whereas for his proofs of some the above elementary results, the relevant euphoria may be shared with million of students and their teachers throughout the world, spiritually, without any kind of connections whatsoever with them. In fact, my motivation for writing this paper is to revisit some of these elementary results and give some more comments about them and reveal the importance and the novelty of Karamzadeh's methods in the proofs of some of these results (note, as I have already made it clear in [29, Acknowledgments], I have learned many interesting stories in mathematics (including facts and proofs in these stories) from him. I must also make it clear that Karamzadeh has already dealt with many of these results, see [27], [7] and in his published articles in some other appropriate journals. However, I am going to deal only with a few of them, due to the scope of this article. Naturally, this is somehow selective and personal. Fortunately, my choices are not limited and I do my best that these choices of results to be in the reader's interest. In particular, in this article I prefer to deal with those elementary results which are quite eligible to appear in the appropriate textbooks for the courses which are being taught to our students at school. Fortunately, contrary to the style of writing research articles in any particular field, there is no need for any preparatory work in this case, for this article is to be read and enjoyed by anyone interested in mathematics. Consequently, without further ado we begin our journey through Karamzadeh's notes on the aforementioned results and present our comments and thoughts on some of them.

2. **Deus ex machina vs. Karamzadeh's rational thoughts**

Paul R. Halmos in [9, p.22] comments that the numbers 2 and $\frac{1}{2}$ are both rational numbers but $2^{\frac{1}{2}}$ is not. This means there exist rational numbers a and b such that a^b is irrational. He then asks if that can be done the other way round. Hence, this is officially put as the following problem in his book. Incidentally, this problem and its solution had already been dealt with, at least 14 years earlier than in Halmos's book , see [18] and [6].

Problem. [9, Problem 3B.] Do there exist irrational numbers α and β such that α^β is rational? That is to say does there exist a rational number with an apparently irrational representation such as α^β?

For the solution, Halmos argues that if $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done, for just put $\alpha = \beta = \sqrt{2}$. If not, then by taking $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$ we are through, without any extra work, see [9, P. 171]. It seems we have not done any mathematical work except invoking the fact that every real number is either rational or irrational. However, Halmos makes it clear to us that by a sophisticated result of Gelfond-Schneider which is a solution of Hilbert's seventh problem, namely, if a and b are algebraic numbers with $a \notin \{0, 1\}$ and b is a real irrational number, then any value of a^b is transcendental, hence it is irrational (e.g., $\sqrt{2}^{\sqrt{2}}$). Consequently, $\sqrt{2}^{\sqrt{2}}$ is indeed irrational. However, one may ask how on earth it occurred to Halmos or anyone else to consider the numbers $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$?

Although Halmos's proof is just fine but it seems more like a narrative which sets up for the punch line of a very good mathematical joke. Karamzadeh in [18] claims rightly that this proof is indeed a deus ex machina. Now let us see what are the rational thoughts of Karamzadeh on the above problem. He suggests the following remarkable theorem, whose proof is as simple as the proof of the irrationality of $\sqrt{2}$, and provides a natural solution to the above problem too, without any need for a deus ex machina or the theorem of Gelfond-Schneider. In particular, it shows that in fact there are infinitely many such α and β, see [27, P.131], for a sketch of proof. In what follows we give a full statement of the theorem with its complete proof for the sake of the reader. And surprisingly we notice that all positive rational numbers except 1 have infinitely many such apparently irrational representation. Let us also imitate Conway and ask why should only Fermat and Conway have little theorems?, see the witty comment of Conway in [18], therefore I also take this opportunity and call this theorem "Karamzadeh's little theorem". Incidentally, Conway's little theorem was first published in 1976 in a joint article of Conway with A. J. Jones as a comment, he republished it again in 2014 apparently with a different proof but this time he calls it his little theorem. Incidentally, Karamzadeh has given a very elementary short proof to this little theorem of Conway too, see [19]. Before stating Karamzadeh's little theorem, let us first consider the original source related to this result. In his famous lecture in 1920, Hilbert proposed 23 problems for the upcoming generation of mathematicians. Indeed, he had prepared 24 problems in his mind for the presentation. However, the last one which was about whether every theorem has a proof that is as simple as it can be, did not appear in the list of the problems, presented in that lecture, see also [18]. It is interesting to notice that Hilbert's seventh problem, i.e., his conjecture, to which Karamzadeh's little theorem is related (note, Euler's conjecture and Hilbert's conjecture, shall be stated briefly) is also connected to the fact which can be stated in a simple geometric language, namely, "if, in an isosceles triangle, the ratio of the base angle to the angle at the vertex be algebraic but not rational, the ratio between base and side is always transcendental". In spite of the simplicity of this statement, Hilbert believed that its proof similarly to the possible proof of his seventh problem would be very difficult. For the latter problem, one may see his conjecture, following that of Euler's, briefly. We should bring to the attention of the reader that Hilbert rightly believed that in dealing with mathematical problems, specialization plays, a still more important part than generalization, see [10, P. 411]. Considering this comment of Hilbert, We admit that Karamzadeh's little theorem can be considered as a very special case of Hilbert's seventh problem or even a consequence of Euler's conjecture. However, we believe the attractiveness of the little theorem's statement and its constructive, simple and natural proof, which is independent of the sophisticated and difficult proof of Gelfond-Schneider of the problem of Hilbert, are what make it stand out as a conspicuous theorem of its own. Incidentally, apparently Karamzadeh has briefly mentioned this little theorem as an observation, in contrast to the above argument of Halmos, which he considered to be a deus ex machina. This observation is first appeared in Karamzadeh's talk at AIMC 25 in 1994, at Sharif University, where he was an invited speaker, see also the next remark.

Theorem 2.1. (Karamzadeh's little theorem (1994)**).** *Every positive rational number, $r \neq 1$ can be written in the form of $r = a^b$, where a and b are irrationals. In particular, there are infinitely many such irrational representations for r.*

Proof. Let $r = \frac{m}{n}$, where $(m, n) = 1$. Now take p to be a prime number with $(m, p) = (n, p) = 1$. Then by the definition of logarithm we have $r = \sqrt{p}^{\log_{\sqrt{p}} r}$. Clearly, $\sqrt{p}$ is irrational. It remains to be shown that $\log_{\sqrt{p}} r$ is also irrational. To this end, let us assume that $\log_{\sqrt{p}} r = \frac{k}{l}$, i.e., $r = \frac{m}{n} = \sqrt{p}^{\frac{k}{l}}$, where k, l are positive integers with $(k, l) = 1$ and seek a contradiction (note, without loss we may assume that $\frac{k}{l} > 0$ for otherwise $\frac{1}{r} = \frac{n}{m} = \sqrt{p}^{\frac{-k}{l}}$, where $-k > 0$, and n would then play exactly the role of m in the proof that follows). Now consider the former equality, i.e., $r = \frac{m}{n} = \sqrt{p}^{\frac{k}{l}}$. Raising the two sides of the latter equality to the power of $2l$ we get $m^{2l} = p^k n^{2l}$. It implies that p must divide m which is the desired contradiction. The proof of the last part is evident for there are infinitely many prime numbers p with $(m, p) = (n, p) = 1$. □

In my opinion the next comments are important from mathematics education points of view. I cannot help concluding this starting part of the article with expressing my strong belief (but, this time with mathematical reasonings for this unshakable belief, which usually needs no proof or evidence) that the above simple and remarkable little theorem of Karamzadeh, together with the above problem in Halmos's book, should be included in our school textbooks, where the irrationality of $\sqrt{2}$ is discussed. In that case, I have no doubt this will give our students deep insights and motivations in learning mathematics. Why not our students see the names of some of our living mathematicians with their original appropriate results?, in some of the textbooks that they are studying for their courses in mathematics to get encouragements, motivations and insights. We should make it clear not all the good mathematicians have appropriate results to be included in elementary textbooks, even among the results of the winners of Fields Medal we might not find any. Without insights and motivations nobody may be willing to devote much time to learning mathematics and doing anything worthwhile in it. For a proof of the latter comment we may read the autobiography of some of the successful mathematicians to see that most of them owe their success to getting some insights after seeing a beautiful result or a non-standard proofs of some tricky results. Mentioning the above problem in Halmos's book and the above little theorem of Karamzadeh in the same course of mathematics, with proper discussions and explanations by the related teacher, will certainly give the necessary insights and motivations to any young student interested in mathematics. And at the same time it can also have a serious impact on those students who are skeptical towards the beauty of mathematics to convince them otherwise. Before, going into the next section, let us recall one of those stories which I learned from Karamzadeh, see [29, Acknowledgment]. Motivated, by his little theorem we might extend the above argument of Halmos to show, in the same vein, that there are in fact infinitely many irrational numbers α, β as in **Problem 3B** (note, Karamzadeh's little theorem has already shown this). To this end, take any two natural numbers $0 \neq a \neq 1$, $0 \neq b \neq 1$ with n dividing b such that they are not the n^{th} power and the m^{th} power of some natural numbers, respectively(note, m, n are integers greater than or equal to 2). Then we may consider $\alpha = \sqrt[n]{a}^{\sqrt[m]{b}}$ and $\beta = \sqrt[m]{b}^{m-1}$ and repeat the same argument above for α and β (noting that also in view of Gelfond-Schneider's result above α is irrational and β is manifestly irrational too, but $\alpha^{\beta} = a^{q}$, where $b = nq$, hence the problem is settled). However, without invoking the result of Gelfond-Schneider, we may also settle the problem by just the above argument which we already called it a good mathematical joke. Karamzadeh used to say that clearly the latter argument generalizes Halmos's, for just put $a = b = n = m = 2$. However, he admitted that we should not get any credit for this observation for there is not any new method in the proof (note, we just created a problem for the sake of the methods that we had already applied before, see the above citation from [37], which we started off our article with). In fact, it looks like repeating a good joke and this time it is just a good mathematical joke. It is time to recall Karamzadeh's motto that "there is a difference between learning mathematics and understanding it", see [36]. A person who uses the above argument and settles the problem by claiming that if $\sqrt{2}^{\sqrt{2}}$ is rational then we are through, otherwise $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ settles the problem has learned this problem well and his or her solution is also fine. However, concerning the irrationality of $\sqrt{2}^{\sqrt{2}}$, if this person is still undecided about it, then he or she has not really understood the real part of the solution. I cannot help contrasting karamzadeh's little theorem with his above generalization of the problem. The former theorem can be followed (i.e. learned) and be understood well, by anyone with a good high school background knowledge, however the latter generalization or even the original problem cannot be understood well without knowing a result similar to the result of Gelfond-Schneider.

Remark 2.2. This remark is added after I watched Karamzadeh's recent talk "Research week, its pros and cons" at Shahid Beheshti Universty during the Research week this year, which was also presented online. My above comments related to Karamzadeh's little theorem, and a point that Karamzadeh brought out in his recent talk (note, he made it clear that the Taxicab number 1729, the smallest number which can be written in two different forms as the sum of two positive cubes, i.e., $1729 = 10^3 + 9^3 = 12^3 + 1^3$ and it is generally known, in the literature as Hardy-Ramanujan number, since 1921, is in fact discovered almost three centuries earlier by the French mathematician Bernard Frènicle de Bessy in 1657). Apparently Bessy send his discovery of the number 1729 in a private letter to John Walis in 1657. This letter was later published in Walis's book "Arithmetria Infinitorum in 1658". These findings made me think it over and have felt that I should dig up more information about the above problem of Halmos. Although Karamzadeh had already noted in [18] that before Halmos, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ was already dealt with in [6], but he did not reveal that whether [6] was the earliest source. Therefore by doing some extra research I noticed that this method of Halmos for dealing with the above problem was first perhaps appeared in [12] and not in [6]. Also before Hilbert's seventh problem concerning the number α^β, where α and β are real numbers, I noticed that it was Euler who first gave the next conjecture and in fact it seems that Hilbert's Conjecture (i.e., his seventh problem) was motivated by Euler's.

Euler's Conjecture (1748). If $a \neq 1$ is a nonzero rational number and b is an irrational algebraic number then a^b is irrational.

Hilbert's Conjecture (1900). If a, b are algebraic numbers with $a \neq 1$ or 0, and b is irrational then a^b is transcendental. Clearly, Hilbert's Conjecture implies that of Euler's. Gelfond and Schneider in 1934-1935, independently resolved the above conjecture known as Hilbert's seventh problem. Consequently, for any positive rational number $r \neq 1$ one can easily get an irrational number $r^{\sqrt{2}}$ and, $(r^{\sqrt{2}})^{\frac{1}{\sqrt{2}}} = r$, i.e., by Gelfond-Schneider theorem any positive rational number $r \neq 1$ can be written as an irrational number to the power of an irrational number. However, if we compare this latter fact with that of Karamzdeh's little theorem, although they prove the same fact, however we must admit that results such as Gelfond-Schneider theorem are very sophisticated and difficult to prove and certainly it will take many years (or, perhaps centuries) to come to be ready in the future for presentation for kids at school. However, Karamzadeh's little Theorem is as simple as the irrationality of $\sqrt{2}$ and it can be inserted in the school textbooks for kids, tomorrow. This is why we believe this interesting observation of Karamzadeh needs to be called his little theorem to attract the attention of the reader and in particular the authors of school textbooks. Invoking some complicated theorems to settle a natural question asked by school kids not only keep them away from mathematics it may also frighten them for ever. For example, if someone asks us why 125 cannot be written as a sum of two nonzero integer cubes and we promptly say because of the Last Theorem of Fermat, have we given a proper solution? By the way, due to the importance of questions related to numbers which are of the form of an irrational number to the power an irrational number, some well-known logicians and mathematicians have published and commented on the subject, see [1, 12, 14, 38].

3. AM-GM Inequality is made as "a walk in the park", by Karamzadeh

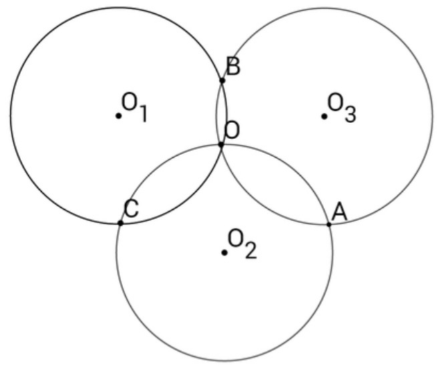
The AM-GM inequality which simply says the arithmetic mean of any finite non-negative real numbers is not less than their geometric mean, i.e., $\frac{x_1+x_2\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n}$, where $x_i \geq 0$ with $i = 1, 2, \cdots, n$, is perhaps the best known and most useful nontrivial inequality in mathematics which generates many other inequalities. Perhaps one may claim it has the largest number of different proofs in the literature. Some of these proofs were given by the first rate mathematicians like, Hölder, Cauchy, Hardy, Polya (just to mention a few) and many others. It is believed that the Greek mathematicians had a geometric proof for the case $n = 2$, see the well-known snakelet (this name is given by Karamzadeh to this possible proof of Greeks), see [15, P. 51]. Let us now deal with Karamzadeh's one line proof of this inequality in [21]. He believes almost all inequalities in mathematics are stated in the form of "if P then Q", which is called *pq*-style by Karamzadeh in [17]. He in fact proves the AM-GM inequality is nothing but a trivial consequence of a hidden meaning of $a \leq x \leq b$, where these are positive real numbers. He observes that if $0 < a \leq b$ and $0 < x$ then $a \leq x \leq b$ if, and only if, $(x-a)(b-x) \geq 0$ or if, and only if, $a+b \geq x+\frac{ab}{x}$. This is manifestly a generalization of AM-GM inequality for the case $n=2$, for just put $x = \sqrt{a}b$ and the equality holds if, and only if, either $x=a$ or $x=b$. We may replace x by any function $f(y,z)$, where $a \leq f(a,b) \leq b$ to get infinitely many nontrivial inequalities such as $a+b \geq f(a,b)+\frac{ab}{f(a,b)}$, see [2] for some special cases of this function f. In particular, by taking $f(a,b) = \sqrt[m+n]{a^m b^n}$, where m, n are positive integers, we get the interesting inequality $a+b \geq \sqrt[m+n]{a^m b^n} + \sqrt[m+n]{a^n b^m}$. I do believe the latter inequality should appear in the elementary textbooks, right after the inequality of $a+b \geq 2ab$, which is the special case of the former inequality where $m=n=1$. My colleague Dr. Azarang has rightly claims with good reasoning in [2] that Karamzadeh's proof is not merely another different simple proof of AM-GM inequality. It has some conspicuous advantages over the other proofs of this inequality in the literature. To be honest and with all due respect to all the authors who have already given some proofs to this inequality, I believe not only the existing proofs but also the future possible proofs of this inequality cannot hold candle to Karamzdeh's proof. Because, as once Karamzadeh has claimed rightly the inequality $a \leq x \leq b$ carries intrinsically a stronger mathematical fact than AM-GM inequality and a fortiori than many other well known and important inequalities which are equivalent to the latter inequality, see for example, [23], [26]. Although as it is claimed in [2], the unprecedented statement $a+b \geq x+\frac{ab}{x}$ above is the whole proof of the AM-GM inequality, however I present the proof for the sake of the completeness, which is just a trivial induction away from the latter statement. Incidentally, the above unprecedented statement is indeed the above hidden meaning of $a \leq x \leq b$ which is pointed out earlier by Karamzadeh. Without loss we may assume that $x_1 \leq x_i$ and $x_2 \geq x_i$ for all $i \leq n$. Put $g^n = x_1 x_2 \cdots x_n$. If all x_i are equal we are done, otherwise $x_1 < g < x_2$ and hence by the above unprecedented statement we have $x_1 + x_2 > g + \frac{x_1 x_2}{g}$. This implies, by induction, that $x_1 + x_2 + \cdots + x_n > g + \frac{x_1 x_2}{g} + x_3 + \cdots + x_n \geq g + (n-1)g = ng$ and we are done. I have to emphasize and bring it to the attention of the reader that the main reason which makes Karamzadeh's proof so effective is because the inequality, $a \leq x \leq b$ is stated in the form of "if, and only if", by him, and it is observed to be equivalent to that useful unprecedented statement which consists of just one inequality, namely, $a+b \geq x+\frac{ab}{x}$. To me Karamzadeh's proof of the AM-GM inequality contains a salient feature, which seems to have been even overlooked, and not emphasized on, in [21] and [2]. It is so conspicuous that I cannot help bringing it up to the attention of the reader, and that is as follows: as we all know we may write $-a \leq x \leq a$ equivalent to a single inequality $|x| \leq a$, where a is a positive real number, conventionally. Surprisingly, Karamzadeh observes that the two inequalities $a \leq x \leq b$, where a, x, b are positive real numbers, is equivalent to the single inequality $a+b \geq x+\frac{ab}{x}$, by thinking mathematically in the form of "if, and only if", and not conventionally, see [17]. This is important from historical and mathematical education perspective, because rarely inequalities are presented in the form of "if, and only if", see [17].

Let us conclude our discussion about this inequality with the comments which follows. We all know that there are not many results in mathematics which receive different proofs from different authors. These results are mainly elementary. One may ask why these authors should bother to give these different proofs to these elementary results? In my opinion it is because certainly these are some basic results and they play important roles in the learning of mathematics and have more chances to be appeared in the future textbooks. Especially, if a proof is simple enough and uses elementary tools it is more likely to get the attention of the younger kids and their teachers at school too. This is naturally a good way for introducing good mathematics to younger generations as early as possible. Again, I do believe the above inequality with Karamzadeh's proof is quite ready to be inserted in a suitable school textbook. Before, going to the next section, we cannot help dealing with an inequality which has recently appeared in one the issues of Quora. This inequality asserts that $\sqrt{\frac{x^2}{y}}+\sqrt{\frac{y^2}{x}} \geq \sqrt{x}+\sqrt{y}$ for any two positive real numbers x, y. The solution in quora is not an elegant one and is settled by a lengthy manipulation. We notice that this is just a very appropriate example which can be settled naturally in a very quick way by Karamzadeh's unprecedented statement, i.e., $a+b \geq x+\frac{ab}{x}$, where $a \leq x \leq b$. Just put $a=\sqrt{\frac{x^2}{y}}$ and $b=\sqrt{\frac{y^2}{x}}$ and assume without loss of generality, that $x \leq y$. Then clearly we have $a \leq \sqrt{x} \leq b$. Consequently $a+b \geq \sqrt{x}+\frac{ab}{\sqrt{x}}$. Clearly $\frac{ab}{\sqrt{x}}=\sqrt{y}$ and we are done, thanks to this amazing unprecedented statement of Karamzadeh.

4. Karamzadeh's three arbitrary circles theorem vs. Johnson's three circles theorem

Let us first quickly recall Johnson's three circles theorem. Roger Johnson in 1916 published a paper in The American Mathematical Monthly under the title "A Circle Theorem" in which he proved if three equal circles pass through the same point, then the circle which passes through their other three points of intersection has also the same radius as these circles. Before rewriting the above theorem with some notations, let us borrow the next comment from [11, P. 4], about this theorem, to see how it is considered to be as an unexpected worthwhile discovery of elementary geometric theorem in the last century. "In twentieth century one does not hold out much hope that there remain to be discovered really pretty theorems at the most elementary level of geometry. However, the American geometer Roger Johnson seems to be the first to come across the above theorem which is within the reach of students at high school taking a first course in Euclidean geometry". Concerning the latter comment, I couldn't help adding the comment that follows which is important from the mathematics historical points of view. Gheorghe Titeica (publishing as Georges Tzitzeica, see [31]) was a great Romanian geometer who enthusiastically worked to help and encourage the school kids in Romania to learn and love geometry. It is said in a meeting in 1908 while he was busy doodling and amused drawing circles with a five lei coin, accidentally discovered a result which was later named Titeica 3 circles problem (5 Lei problem) and proposed it the same year at a competition organized by Romanian Mathematical Gazette. This is the same theorem as Johnson's three circles theorem above which was rediscovered independently by Johnson 8 years later than Titeica, in 1916. Let us, before stating the above theorems in the title, bring to the attention of the reader a saying of Karamzadeh which is related to the above discovery of Titeica, too, namely, " mathematics is well-defined doodling, and this doodling is never complete unless it can be shared by laymen", see [17, footnote].

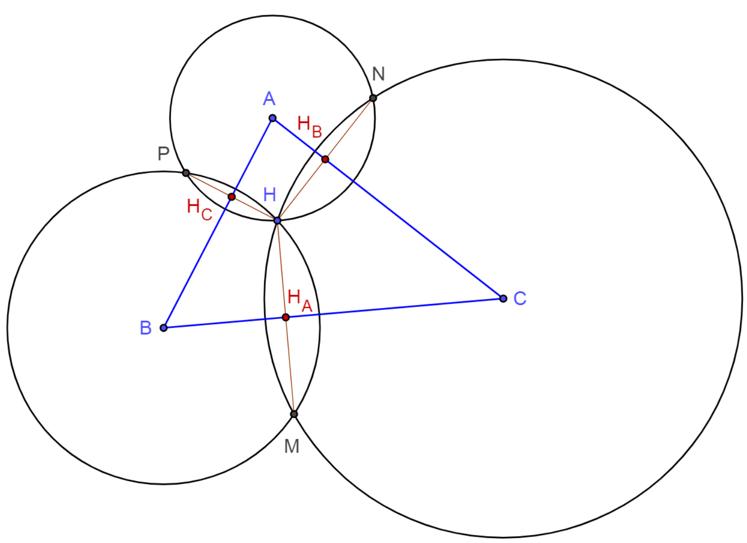
Johnson's three circles theorem. *Let the circles $A(r), B(r)$ and $C(r)$ with centers A, B, C, respectively, and the same radius r pass through a point H and have the second points of intersection M, N, P. Then the circle which passes through the latter three points has the same radius r, too.*



Titeica's 5 Lei problem

There are many different interesting proofs of this theorem in the literature, see [3, 11, 13, 25]. Now let us state Karamzadeh's theorem in relation to the above theorem.

Karamzadeh's three arbitrary circles theorem. *Let three arbitrary circles with centers A, B, C pass through the single point H and intersect pairwise in the points M, N, P. Let H_A, H_B, H_C be the feet of perpendiculars from the point H to the lines BC, AC, AB, respectively. The triangle $\triangle MNP$ is similar to the triangle $H_AH_BH_C$ with the ratio of similarity 2. In particular, if the three circles are equal, then the triangles $\triangle MNP$ and $\triangle ABC$ are congruent and their circumcircles are congruent, with the original three too.*



Before proceeding any further, let us recall the convention that three collinear points form a degenerate triangle. In the case of Johnson's Theorem, $\triangle MNP$ is never degenerate, while in the general case, where the three circles need not be equal, the corresponding triangle may be a degenerate triangle.

Karamzadeh used to claim rightly that Johnson's theorem is a consequence of his theorem and, at the same time, his theorem needs no proof at all. However he admits that the proof in [16] was given apparently because of the editor's demand for a proof for the sake of a reader who might be a very young student and a bit wet behind the ears about circles. What follows is essentially that straightforward proof. Clearly, H_A, H_B, H_C are the midpoints of HM, HN, and HP, respectively. Hence we have $H_AH_B = \frac{1}{2}MN$, $H_AH_C = \frac{1}{2}MP$ and $H_BH_C = \frac{1}{2}PN$, i.e., the triangle $\triangle MNP$ is similar to the triangle $\triangle H_AH_BH_C$ with the similarity ratio equal to 2 (clearly, if any of the latter triangles is degenerate so too is the other one). Now if the three circles are equal then manifestly the triangle $\triangle H_AH_BH_C$ is the medial triangle of triangle $\triangle ABC$. This means that both triangles $\triangle ABC$ and $\triangle MNP$ are twice as large as the triangle $H_AH_BH_C$, i.e., the former two triangles are congruent which in turn it implies that their circumcircles are congruent too. Clearly the circumcircle of the triangle $\triangle ABC$ has the point H as its center, i.e., its radius as well as that of the circumcircle of the triangle $\triangle MNP$ is the same as the original three and hence we are done.

Although it seems we have written a few lines for its proof, however as Karamzadeh once admitted the proof of his theorem is nothing but using the classical and trivial fact belonging to the era of Euclid, namely, if two circles intersect in two points, then the line through their centers is the perpendicular bisector of the common chord. He also believes that when a fact is not stated in its general form, usually one might be misled toward finding a proper proof. For example, one may compare the trivial and natural proof of Karamzadeh's theorem above with various proofs of Johnson's theorem in the literature to notice the difference, see the proofs in the list of references in [16] and, in particular, the first one in [11, P. 18] and the proof in [33, Chapter 10] which are in the same vein. Let us examine more closely some part of the latter proof for its excellent ideas and explanations which matters a lot in problem solving and in mathematics in general. Polya for introducing "The coming of the idea" chooses Johnson's theorem for his discussion on how to get an idea for its proof. He tries to reconstruct a sequence of excellent ideas that led to its proof. In the configurations, in which he draws, he encounters many rhombi (note, these are also called rhombuses or equilateral quadrilaterals). He even encounters with the projection of of the 12 edges of a parallelepiped and admits that the theorem is proved surprisingly by artistic conception of a plane figure as the projection of a solid (i.e., the proof uses notions of solid geometry). He then hopes that this is not a great mistake, otherwise it is easily redressed. We believe in that case it might be redressed easily and also we admit that he presented very attractive personal view points and rare ideas before reaching the final stage of the proof. I must confess I enjoyed most of his comments in [33, Chapter 10], especially his comment that says, ideas come when they want to come, not when we want them to come, and waiting for ideas, is gambling. However, in my opinion, with all due respect to him (note, Polya is admired by many authors, including myself) his choice of taking Johnson's theorem as an example to discuss and present his great ideas for its proof, was not perhaps a good choice. Because as we noted above the proof of this theorem needed only simple ideas belonging to the era of Euclid, i.e., no fuss, no muss. Finally, we couldn't help concluding our discussion, about Johnson's theorem, with recalling the next interesting consequence of this theorem, see [11]. If we consider the incircle of a triangle $\triangle ABC$ as the circle of an inversion (note, its center, i.e., the incenter of the triangle is taken to be the center of this inversion and the square of the inradius is taken to be the constant of the inversion). Then clearly under this inversion each side goes into a circle which passes through I, the incenter of the triangle and is tangent to both the side and the incircle of the triangle. Hence the three inversions of the sides, which are three circles, pass through I and have the same radius, because their diameters are the inradius of the triangle. Hence these three circles satisfy the first part of the statement of Johnson's three circles theorem. Now since the circumcircle of the triangle goes through vertices and each vertex lies on two sides we infer the inversion of the circumcircle must be the circle which goes through the intersection points of the latter three circles except I. In sum, we may say the four circles which are the inversions of the sides and the circumcircle of the triangle have the same radius. One last word: if we carefully draw this triangle with all these mentioned circles then we can naturally see the verity of Johnson's three circle theorem in this configuration, i.e., as the old adage rightly says, a picture is worth a thousand words. This can also be considered as a visual proof of Johnson's three circles theorem.

5. The full story of the shortest possible solution in the history of the solutions of the official problems of the IMOs and Karamzadeh's role in that solution

Let me first give my reasons for dealing with the above topic in this article. I have to first emphasize that the above ordinary problem which has received the shortest possible solution so far among all the IMO problems from the beginning of the International Mathematical Olympiad (i.e., from 1st IMO 1954 Romania until now) by no means is comparable with other classical problems or results in this article, as far as mathematics is concerned. However, in the next few lines I am going to write about it for the sake of the oral history of mathematics in our country and as the title shows it is somehow related to Karamzadeh too. Before writing this article and some of my recent articles, see [28] and [29], I had to read [7], [27] and some other articles by Karamadeh. In [7, P. 51], I encountered a peculiar title "The shortest proof in the history of IMO". Although I had read the book before, but this time I had to pay a close attention to the details. That section of the book starts with a problem which was among the six problems of, 32nd IMO 1991, held in (Sigtuna, Sweden). Incidentally, the problem which was proposed by France, is essentially as follows (note, although the statement of the problem in [7], is more general but without referring it to Figure 1, it is evident, however, it is correctly stated here). Take a point inside a triangle $\triangle ABC$ and join it to the vertices of the triangle. Show that among the six angles formed at the vertices, there are two of them (i.e., one among the angles $\alpha_1, \beta_1, \gamma_1$, and the other one among the three other angles, in Figure 1) which are less than or equal to $30°$, see [7, P. 51-53], to see a short story about this problem and how might the statement of this problem is guessed. Let me digress for a moment and recall what follows from a recent conversation with Karamzadeh concerning the previous peculiar title (note, as I mentioned above I wanted to learn the details). The original solution of the problem was a three page solution. At first, as Karamzadeh remembers, this was a drawback for the problem to be selected by the Jury, consisting of the

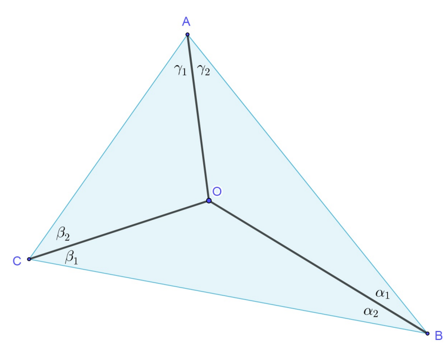


FIGURE 1

team leaders of the participating countries, as one the six problems for that contest. He said: we and some other leaders were interested in that problem for the sake of our students' tastes for geometric questions. Therefore, before the final meeting of the jury, we had a private meeting with those leaders who were pro geometric problems and discussed our strategy for the final jury meeting. However, apparently there was an unwritten rule which could increase the possibility of the selection of the problem in the final meeting of the jury, with such a drawback. And that is, if anyone of the leaders or deputy leaders can give a proper and shorter solution before making the final decision, by the jury, then this might help for that particular problem to be selected as one of the final six problems. In [27, P. 52], Karamzadeh claims that he had found such a short solution which is less than of half of page and this helped the problem to be finally selected. He also claims that the team leader of Spain, F. B. Rosado, who was a member of that private meeting, has co-authored a book containing some problems in Combinatorics, Algebra, and Geometry, has inserted his solution in that book and has sent him a copy of the book, with a written kind message inside the book. But unfortunately because of having a disorderly library he couldn't find the book in his library, see [7, P. 52]. Suddenly, it occurred to me the idea that I could suggest my help for finding that book in his library. I did that and he happily accepted. But while laughing he said it would be like "looking for a needle in a haystack", because books and journals are scattered everywhere in my library even there are some on the ground, on the bed, under my pillows, on the sofas, etc. I said OK, I can manage it by myself. Anyway, one day I went to his place and after two or three hours of searching, I found the book on a shelf among the many old issues of American Mathematical Monthly and Mathematical Gazette. Now before dealing with the solution of this problem, let me first recall the next two theorems which are related to the solutions of the problem and also to my next comments.

Theorem 5.1. *(Erdös-Mordel Theorem). If O is a point inside a triangle $\triangle ABC$ and x, y and z are the lengths of three perpendiculars to the sides of the triangle, then $OA + OB + OC \geq 2(x + y + z)$.*

Theorem 5.2. *If a, b, c and S are the side lengths and the value of the area of a triangle $\triangle ABC$, then $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$.*

This theorem is apparently discovered by Vitzenbeck in Math.Z. 1919, and then in 28 years later rediscovered by Finsler and Hadwiger in Comm. Math. Hel, see [27, P. 84], to see different proofs of this inequality and how mathematicians invent inequalities. Incidentally, in the first IMO 1954, in Romania, this theorem was among the six problems of the contest. Whereas the problems which are to be presented as questions at any contest for the evaluation of the contestants are supposed to be original, otherwise it wouldn't be a fair contest. Karamzadeh who has given the latter comments also claims non-original problems are appeared among the six problems of some later IMO contests, too. Karamzadeh used to introduce almost all the important theorems and problems in elementary mathematics in his training classes for the selected IMO students of our country for many consecutive years, including the above theorems. None of these results have ever appeared, in the Farsi literature, before his introduction of these results in those classes. Of course, some of these results have later appeared in some books and articles in Farsi. I just wanted to remind the reader and emphasize the importance of the role of those training sessions for the IMO students not only for the preparation of those limited number of students for the IMO, it has also affected and promoted our mathematics literature in Farsi, too. Therefore one thing that I have learned about him, which I did not notice before, while preparing this article, even at the time that I wrote [28], is the latter important fact. This also persuades me to recall another point about him that I did not mention in [28]. I should also admit that when one is listening to him talking about any result in mathematics, he recalls some interesting anecdotes related to the result so enthusiastically that you wouldn't notice the passing of the time. In short, his enthusiasm for mathematics is so strong that is contagious. For example, see his article "generalization in mathematics" in [27]. Let us now cite his solution of the above problem (IMO 1991), from [35, P. 115]. Let P to be point inside an acute triangle $\triangle ABC$. Put $\angle PAB = \alpha$, $\angle PBC = \beta$ and $\angle PCA = \gamma$. Also put $x = PA$, $y = PB$ and $z = PC$ and assume that none of the angles α, β and γ are less than or equal to 30° and seek a contradiction. Let S be the area of this triangle. Then we may write $S = \frac{1}{2}(cxsin\alpha + aysin\beta + bzsin\gamma)$ and notice that since all these angles are greater than 30°, we immediately infer that $S > \frac{1}{4}(cx + ay + bz)$. Consequently, in view of the second theorem above we have $a^2 + b^2 + c^2 > \sqrt{3}(cx + ay + bz)$. Also since $x^2 = b^2 + z^2 - 2bzcos\gamma$ we infer that $x^2 > b^2 + z^2 - \sqrt{3}bz$ (note, $cos\gamma < \frac{\sqrt{3}}{2}$). Similarly, we have $y^2 > c^2 + x^2 - \sqrt{3}cx$ and $z^2 > a^2 + y^2 - \sqrt{3}ay$ and by these three latter inequalities we get $a^2 + b^2 + c^2 < \sqrt{3}(cx + ay + bz)$, which is the desired contradiction. Next, let us recall Shahram Mohsinipour's solution, who was as a student a team member of our country then, see [7, P. 53]. What he claimed as his proof was just this: This is obvious by Erdös-Mordel Theorem. Karamzadeh once said when we presented this solution to the Swedish. coordinators, who were responsible for the evaluation of the students' solutions and for giving marks for this particular problem, were waiting to see any more details in Shahram's solution of the problem. When I explained what shahram means with those few words, they said OK, but how can we be certain that there is such a theorem. I claimed I myself used to mention this theorem in my classes during their training sessions in Iran, before coming to Sweden. They argued and said sorry that is not a reliable source for us, we need published books or journals where that theorem is proved. I said OK, you may search for it in Coxter's book, Introduction to Geometry, John Wiley. They said OK we do that and we will let you know about our final decision tomorrow. The next day one of them came to us admitting that you were right and Shahram will get the full mark for his solution. Although Rosado, the team leader of Spain in his kind message inside the book calls Karamzadeh as one of the stars of his book (see Figure 3), but Karamzadeh modestly admits that Rosado was not aware of Shahram's solution then, otherwise he would have also included Shahram's proof in that book and certainly would mention his name as the "superstar of the book". When I later argued with Karamzadeh that if one is going to write down Shahrama's solution in detail then we have to write down similar inequalities and then more or less the two proofs would be of the same length. He said but we have to notice that in [7, Section 2], I am talking about "unforgettable proofs", when one says by Erdös-Mordel Theorem every thing is clear, it means writing down those necessary inequalities are natural and therefore Shahram's proof is indeed, unforgettable, but although my proof might be of the same length however writing the necessary inequalities to complete the proof is, in no ways, as natural as those in that of Shahram's. This is why my proof is not unforgettable as that of Shahram's. I repeated, but anyway you had both a direct and an indirect role in that shortest solution because, after all it was you who introduced Erdös-Mordel theorem and many other important results to our selected IMO students and, in particular, to Shahram, in their training classes. While was walking away with a smile on his face, he shrug it off as though it was his natural duty. Later, he said but we should all admit, by invoking Erdös-Mordel Theorem for solving that particular problem by Shahram, the statement of that problem may be recorded naturally as an immediate corollary of the theorem.

6. A century old mystery of Morley's Theorem is finally resolved by Karamzadeh!

Let us first recall a statement for the above theorem. Let the adjacent trisectors of a triangle $\triangle ABC$ meet at the points X, Y, Z as in Figure 2(B). Then the triangle $\triangle XYZ$ is called the Morley's triangle of triangle $\triangle ABC$. Frank Morley an English-American mathematician while working on a problem in algebraic geometry came across the following incredible theorem, see [20].

Morley's Theorem (1899). The Morley's triangle of every triangle is equilateral.

My motivation for writing this part, in the first pace, is to claim with emphasis and without any hesitation that Karamzadeh has transformed Morley's Theorem, which was notorious in the last century for being mysterious, into such a simple theorem that can be presented to our school kids with least possible elementary tools, so that any other way for presenting it may use less elementary tools. Also I have recently came across a comment in a new book [32] which rebukes a claim by, M. Lange, a well-known logician, in [22, p. 253]. Lange while studying Morley's Theorem notes how this theorem is often called "Morley's Mystery" or "Morley's Miracle", even though it has been re-proved over and over again many times in many different ways by some first rate mathematicians. For Lange the mystery, as claimed in [32], tied to there being no explanatory proof of the theorem. However, the author in [32], in order to refute this claim of Lange, says at least one mathematician, namely, Karamzadeh, has shown that there exists a proof which is completely explanatory, see [32, p. 41-43]. There is also another author, see [5, p. 84], who admires this explanatory proof of Karamzadeh, too. Indeed, the author in [32], emphasizes that the reason why Karamzadeh has succeeded resolving this so-called mystery in the theorem is because with the help of his Bisector proposition, which will be stated, shortly, see also [18]), he has stated Morley's Theorem for the first time in the history of this theorem, see [18], in the form of "if, and only if". I must bring to the attention of the reader that stating the results in the form of "if, and only if" was one of the main reasons that Karamzadeh got attracted to the subject of $C(X)$, see [29, p. 155]). The author of [32, p. 43], continues to say rightly, more generally we can see how this sort of biconditional (i.e., the statement of the theorem in the form of "if, and only if") allows for its explanatory proof. Let us before going any further recall Karamzadeh's Bisector Proposition which seems to be have been overlooked since Euclid time, for its natural proof see [18].

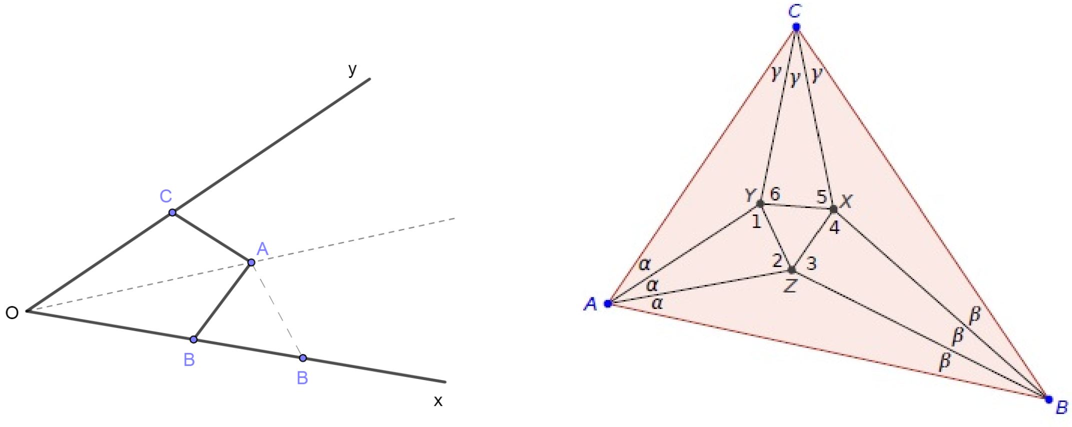
Karamzadeh's Bisector Proposition. *Suppose that A is a point inside the angle $\angle xOy$ and B, C are two points on the arms Ox and Oy, respectively* (Figure 2(A)). *Then if any two of the following hold, so does the third.*

(1) A lies on the bisector of the angle $\angle xOy$.

(2) $AB = AC$.

(3) Angles $\angle OBA$ and $\angle OCA$ are either equal or supplementary.

The above proposition is in fact in the form of "if, and only if", for (1),(2) hold if, and only if, (1),(3) hold, or if, and only if, (2),(3) hold. Before the publication of Karamzadeh's articles related to Morley's Theorem, see [20], [18] and [8], many authors believed that among the existing proofs of this theorem the proofs of John Conway and that of D.J. Newman were the simplest ones, especially the former one. I have to remind the reader that the Newman's proof uses trigonometry and he somehow expressed his feeling that he was not happy with his own proof as well as with the other existing proofs of the theorem. As with Conway's proof, there is some unexplained element in the proof, namely, introducing the angles $\angle 1 = \angle 4 = 60 + \gamma$, $\angle 2 = \angle 5 = 60 + \beta$, and $\angle 3 = \angle 6 = 60 + \alpha$, without any explanations, see Figure 2(B). Incidentally, Newman's proof applies these angles also without any explanations. Karamzadeh believes that the latter two authors had perhaps obtained the values of these angles via trigonometry without mentioning it. We may bring to the attention of the reader that there were already proofs prior to Conway's and Newman's whose authors had obtained the values of the above angles by trigonometry, see for example, Banhoff's proof in `https://www.cut-the-knot.org/triangle/Morley/BankoffProof.shtml`. Incidentally, most of the backwards proofs of Morley's Theorem make use of the above values without any explanations, for example see B. Bollobas's proof in the latter reference and also that of Roger Penrose's, see the



(A) (B)

Figure 2

reference list of [20]. It was, indeed, Karamzadeh who for the first time by invoking his Bisector Proposition geometrically proved that Conway's assignment of the above three pairs of angles in his proof and therefore in other similar proofs are inevitable and quite explanatory, see [18]. Let us go back to Conway and Newman and make some comments about the simplicity of their proofs. By modifying Conway's proof, Karamzadeh in [18], claims rightly that this modified proof which eschews the similarity is clearly simpler than Conway's, however at the same time he admits that this modified proof should also be called Conway's proof. And he emphasizes that this modified proof is perhaps the simplest possible proof for the time being at that time. However, he also predicts and reminds the reader that one cannot be sure if the proof remains the simplest for a long time in the future. His resourcefulness in this regard paid off and it did not take before long that the latter prediction came true. Indeed, Karamzadeh in [20], has manifestly shown with some reasoning that the proof in [8], is the simplest possible and remains so in the literature for ever. Incidentally, his Bisector Proposition enables him to present Morley's Theorem in the form of "if, and only if" for the first time in the history of this theorem. This makes the proof of this theorem quite explanatory and at the same time also reveals what is the possible mystery in this theorem and shows that, again by using Bisector Proposition, the theorem is no longer mysterious, i.e., the mystery is resolved. Due to the significance of the revealment of this mystery which is clearly explained in [20], we also re-emphasize on it by recalling this mystery. By considering the configuration above we notice that the values of angles of triangles $\triangle CYX$, $\triangle AZY$ and $\triangle XYZ$ are determined without any work. If we assume the Morley's triangle in the configuration is equilateral then it remains to find the values of the angles of the other three triangles inside the original triangle $\triangle ABC$, in the above configuration. If we can find the values of angles $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$, we are through. Similarly to [20, P. 300] one can easily write down the next six equations. $\angle 1 + \angle 2 = 180 - \alpha$, $\angle 2 + \angle 3 = 180 - \gamma$, $\angle 3 + \angle 4 = 180 - \beta$, $\angle 4 + \angle 5 = 180 - \alpha$, $\angle 5 + \angle 6 = 180 - \gamma$ and $\angle 6 + \angle 1 = 180 - \beta$. Manifestly, these equations are not independent and any one of these equations may be deduced from the other five. This means in fact we have five equations with six unknowns which usually have not a unique solution. However, naturally the values of these unknowns must be unique. Hence we need at least another independent equation. Unfortunately, this cannot be easily inferred from the above configuration. This is where the hidden fact, of Bisector Proposition, in the configuration enters the scene and provide us with extra information, namely, $\angle 1 = \angle 4$, $\angle 2 = \angle 5$ and $\angle 3 = \angle 6$. Consequently, the later three pairs of equalities hold if, and only if, the Morley's triangle $\triangle XYZ$ is equilateral, thanks to Bisector Proposition. Incidentally, this observation together with Bisector Proposition provide the simplest possible proof of Morley's Theorem in [8], which as claimed above remains the simplest for ever, see [20] for details. And as Karamzadeh has already emphasized in [20], the hidden fact, of Bisector Proposition, in the above configuration is the mystery of Morley's triangle. It seems we had to wait all these years for this elegant Bisector Proposition, which has been overlooked since Euclid time, to appear, to provide for us a manifest short geometric proof of Morley's theorem and at the same time to resolve its so-called mystery, too. As for Newnan's proof, first we prefer to quote his useful and honest comments about Morley's Theorem prior to his proof in https://www.cut-the-knot.org/triangle/Morley/newman.shtml. "One of the sad things about the current philosophy of mathematical education is the avoidance of plane geometry. Today's generation, and perhaps their parents as well have not heard of marvels like the 9-point circle, Descartes's theorem, Ceva's Theorem, or the marvel of marvels, the Morley's triangle. As shown in Figure 1, one takes an arbitrary triangle and trisects its angles, obtaining three intersection points. These form the small triangle inside the starting triangle. This small interior triangle is far from being arbitrary, however, Morley's great discovery (1899) being that it is always equilateral! When I read, or rather tried to read, Morley's proof of this startling theorem, I found it absolutely impenetrable. I told myself that maybe in future years I would return and then understand it. I never succeeded in that, and even when I read the much simpler proof based on trigonometry, or the fairly simple geometric proof due to M. T. Naraniengar (1907), there was still too much complexity and lack of motivation. (A series of lucky breaks!) Were we to give up, forever, understanding the Morley Miracle? Or are we failing because we are asking too little? After all, Morley's theorem states that in Figure 1, the inner triangle always will be equilateral. The reason that all the proofs seem to be so difficult and unmotivated is probably because Morley's theorem is really only half the story. The full picture is in Figure 1 and this tells the whole story and indeed proves itself! (This happens often in induction proofs: The fuller statement is easier to prove than the restricted one.)" (note, Figure 1, in the previous comment is the figure in his proof in https://www.cut-the-knot.org/triangle/Morley/newman.shtml). Newman was a great problem solver and his very elementary proof of the prime number theorem is appreciated by many mathematicians, (note, I learned this from Karamzadeh, see [27, Acknowledgments]). By considering Newman's comments about Morley's Theorem, one can surely claim if people like him and geometry lovers like Coxeter were alive today, they would be so happy to see that the mystery of Morley's Theorem is finally resolved. I would like to take this opportunity and conclude my article with the following reminiscence from[4], about Newman. Before quoting this reminiscence, about Newman, I like to recall an anecdote which is also related to Newman. Before Karamzadeh's retirement, once during our regular slow walking together in the campus of our university, he was recalling the next peculiar problem (note, since he was aware of my interest in set theory his usual choice for these kind of problems was in set theory). Let $f(x, y)$ be a function from $\mathbb{R}^2$ into $\mathbb{R}^2$. Suppose that this function is a polynomial in x for each fixed y, and it is a polynomial in y for each fixed x, show that $f(x, y)$ must be a polynomial in x and y. He said that this problem is due to D.J. Newman, who has also given a very elementary proof to the prime numbers theorem. He continued by saying that he was a great problem solver like Erdös (note, that was the first time I heard of the name D.J. Newman and about his elementary proof of the prime numbers theorem). From among rather many of the Newman's colleagues' and his student's reminiscences including John Nash's, see [4], I have chosen Doron Zeilberger's on purpose, which is related to Newman's proof to

Morley's Theorem, too. "Don Newman was a great mathematician, but he was even a greater problem-solver. Problem solving is not the same as math, and I am sure that if Don would have been less addicted to problem-solving, he would have achieved much more in "regular" mathematics, but of course, he didn't care; he just wanted to have fun. To cite just a few of his masterpieces, his proof of Morley's Theorem and his solution of the 12-coin problem are masterpieces worth many "serious" theorems and proofs. I met Don for the first time when I gave an interview talk at Temple, at the beginning of 1990. To illustrate my talk, I put on the blackboard a recent Monthly problem that was meant to illustrate $\sum_{n=0}^{\infty} \frac{1}{n!(n^4+n^2+1)} = \frac{e}{2}$ but Don stole my thunder: he did it on the spot, in less than a minute. Don was not big on e-mail, so often I got e-mail from John Nash, who really adored Don, to print out and give to Don".

One last word regarding Karamzadeh's work on Morley's Theorem: One may certainly claim that even though this theorem is in the list of 100 most important theorems in mathematics, see [20], however is not studied at school in any country officially yet, see the above comments of Newman. Now as we notice, Karamzadeh has transformed this theorem in a way that it can be easily taught to our students at school in their early ages. Why not inserting it in a proper geometry course at school in Iran with Karamzade's approach? To be the first country in the world for doing that. Of course I must inform the reader that, his work is already appeared in a geometry textbook in Farsi, see [30]. However due to its history, it would be more appropriate if it is also a part of our geometry course at school for a record, where also the students naturally encounter the concept of trisectors (note, Morley's Theorem is perhaps the only theorem in mathematics that deals with trisectors and it is also a proper theorem in which the question of impossibility of trisecting, using a ruler and compass only, of any given angle might naturally be brought up for the students) .

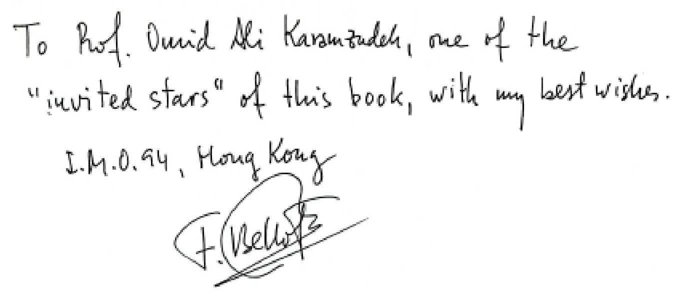


FIGURE 3. (*The handwritten note of F. B. Rosado on the first page of his book)*

Acknowledgments

I would like to express my heartfelt and deep gratitude for all the scientific and moral teachings I have learned from Professor O.A.S. Karamzadeh during every moment of the past 35 years. Undoubtedly, without those teachings, it would not be possible to prepare this article. I would also like to express my gratitude to the meticulous referees for carefully reading this article and for their valuable suggestions. In particular, I am indebted to the referee who brought my attention to a Math bit note, written by Nick Lord (see [24]). It is both quite surprising and causing euphoria that in [24], "the statement of Karamzadeh's little theorem" is guessed without knowing of [27, P. 131], after a span of 14 years, and the author's constructive and simple method in [24] (or, in a way, indirectly Karamzadeh's rational thoughts, and his shooting with plastic bow and arrow, i.e, his little theorem's proof) is metaphorically, contrasted to invoking a big gun of Gelfond-Schneider for the proof of this little theorem. I should also appreciate the latter comment of Nick Lord in [24] for emphasizing on the inappropriateness of using the theorem of Gelfond-Shneider for the proof in this case. I would also like to thank Professor Azarpanah, the editor, for his general comments that made me, in particular, improve the presentation of Section 2.

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