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Journal of Advanced Mathematical Modeling
ISSN (print): 2251-8088, ISSN (on-line): 2645-6141
Vol. 13, No. 5 (2024), pp. 99-107.
DOI: 10.22055/JAMM.2024.45312.2227

MODULES SATISFYING DOUBLE CHAIN CONDITION ON UNCOUNTABLY GENERATED SUBMODULES

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Communicated by: O.A.S. Karamzadeh

Abstract. In this article, we study modules that satisfy the double infinite chain condition on uncountably generated submodules, briefly called $u.c.g.-DICC$ modules. We show that if a quotient finite dimensional module M satisfies the double infinite chain condition on uncountably generated submodules, then it has Krull dimension. We study submodules N of a module M such that whenever $\frac{M}{N}$ satisfies the double infinite chain condition so does M. Moreover, we observe that an α-atomic module, where $\alpha > \omega_1$ is an ordinal number, satisfies the previous chain condition if and only if it satisfies the descending chain condition on uncountably generated submodules.

1. Introduction

The double infinite chain condition was introduced by Contessa for modules over commutative rings (briefly $DICC$-modules); see [4, 5, 6]. Osofsky [18] extended the concept of $DICC$ to objects in $AB5$ category. She characterized $DICC$ objects in this category and obtained some noncommutative generalizations. Karamzadeh and Motamedi [13] undertook a systematic study of the concept of $\alpha - DICC$ modules. Later, Rahimpour [19] studied modules that satisfy the double infinite chain condition on finitely generated submodules, denoted by $f.g. - DICC$-modules. Davoudian [10] studied modules that

MSC(2020): 00A09, 54C40.

Keywords: Uncountably generated modules, Krull dimension, DICC-modules, u.c.g.-DICC modules

Received: 19 November 2023, Accepted: 2 May 2024.

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satisfy the double infinite chain condition on nonfinitely generated submodules, denoted by $n.f.g.-DICC$-modules. An R-module M is called countably generated if there exists a countable subset A of M such that $< A >= M$; otherwise, M is uncountably generated. We extensively studied modules with the chain condition on uncountably generated submodules, see [9]. In this article we study modules that satisfy the double infinite chain condition on uncountably generated submodules, briefly called $u.c.g.-DICC$ modules. We show that if a quotient finite dimensional module M satisfies the double infinite chain condition on uncountably generated submodules, then it has Krull dimension. We also observe that if N is of finite length submodule of M and $\frac{M}{N}$ is an $u.c.g.-DICC$ module, then so is M. If an R-module M has the Noetherian dimension and α is an ordinal number, then M is called α-atomic if $n\text{-}\dim M = \alpha$ and $n\text{-}\dim N < \alpha$ for all proper submodules N of M. An R-module M is called atomic if M is α-atomic for some ordinal α; see [14] (note, atomic modules are also called conotable, dual critical, and N-critical in some other articles; see for example [17, 1] and [3]). We also observe that an α-atomic R-module M is $u.c.g.-DICC$ if and only if M satisfies the descending chain condition on uncountably generated submodules, where ω_1 is the first uncountable ordinal number and $\alpha > \omega_1$ is an ordinal number. Throughout this paper R will always denote an associative ring with a nonzero identity and M a unital R-module. The notation $N \subseteq M$ (resp. $N \subset M$) means that N is a submodule (resp. proper submodule) of M. The reader is referred to [2, 12, 13, 14], for definitions, concepts, and the necessary background not explicitly given here.

2. Preliminaries

In this section we recall some useful facts about modules with Krull dimension and modules with chain condition on uncountably generated submodules. First we recall that, the concept of Krull dimension of an R-module M, denoted by $k\text{-}\dim M$, is the deviation of the poset of all submodules of M. The codiviation of the poset of all submodules of M is called Noetherian dimension and denoted by $n\text{-}\dim M$.

Let us continue with the following well-known and important result; see [16, Corollary 6] or [14, Proposition 1.1].

Proposition 2.1. *An R-module has Noetherian dimension if and only if it has Krull dimension.*

We need the following result which is also in [14].

Proposition 2.2. *Let M ba an R-module. If each proper submodule N of M has Noetherian dimension, then so does M and $n\text{-}\dim M \leq \sup\{(n\text{-}\dim N)+1 : N \text{ is a proper submodule of } M\}$.*

The proof of the next result is similar to the proof of its dual result in [8, Lemma 1.4].

Proposition 2.3. *If M is an R-module and for each submodule N of M, either N or $\frac{M}{N}$ has Krull dimension, then so does M.*

It is well known and easy to see that an R-module M satisfies the ascending chain condition (ACC) on finitely generated submodules if and only if M is Noetherian. Dually, M satisfies the descending chain condition (DCC) on finitely generated submodules if and only if M is a perfect module. We studied modules with chain condition on nonfinitely generated submodules, see [7]. An R-module M is called countably generated if there exists a countable subset A of M such that $< A >= M$; otherwise, M is uncountably generated. In [9], we characterize modules M which satisfy the ascending (resp., descending) chain condition on uncountably generated submodules (i.e., for any ascending (resp., descending) chain $N_0 \subseteq N_1 \subseteq N_2 \subseteq ...$ (resp., $N_0 \supseteq N_1 \supseteq N_2 \supseteq ...$) of uncountably generated submodules of M, there exists an integer n such that for each $i \geq n$, $N_i = N_{i+1}$. In [15], it is shown that every submodule of module with a countable Noetherian dimension is countably generated. Hence, if M has a countable Noetherian dimension or ω_1-atomic, then M satisfies the ascending chain condition (descending chain condition) on uncountably generated submodules.

We recall that the Goldie dimension of an R-module M, denoted by $G\text{-}\dim M$ is the supremum λ of all cardinals k such that M contains the direct sum of k nonzero submodules. Given a cardinal number k, we say k is attained in M if M contains a direct sum of k nonzero submodules, see [11]. We also recall that by a quotient finite dimensional module M we mean for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension.

We cite the following results from [9].

Proposition 2.4. *Let M be a quotient finite dimensional module. If M satisfies the ascending chain condition on uncountably generated submodules, then $n\text{-}\dim M \leq \omega_1$, where ω_1 is the first uncountable ordinal number.*

Proof. See [9, Proposition 3.2]. ∎

Proposition 2.5. *Let M be a quotient finite dimensional module. If M satisfies the descending chain condition on uncountably generated submodules, then it has Krull dimension.*

Proof. See [9, Proposition 4.1]. □

3. Double chain condition on uncountably generated submodules

In this section, we study modules that satisfy the double infinite chain condition on uncountably generated submodules, briefly called $u.c.g.-DICC$ modules. Next, we give our definition of $u.c.g.-DICC$-modules.

Definition 3.1. An R-module M is said to be $u.c.g.-DICC$, if given any doubly infinite chain

$$... \subset M_{-2} \subset M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset ...$$

of uncountably generated submodules of M, there exists an integer k, such that $M_i = M_{i+1}$ for each $i \geq k$ or $M_i = M_{i+1}$ for each $i \leq k$.

We continue with the following lemma, whose proof is given for the sake of completeness.

Lemma 3.2. *If M is an u.c.g.-DICC module, then given any infinite descending chain $N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq ...$ of uncountably generated submodules of M either $\frac{N_i}{N_{i+1}}$ satisfies the ascending chain condition on uncountably generated submodules for all i or there exists an integer k such that $N_{i+1} = N_i$ for each $i \geq k$.*

Proof. Let $\frac{N_r}{N_{r+1}}$ do not satisfy the ascending chain condition on uncountably generated submodules, for some r. Thus there exists an infinite chain $\frac{N'_1}{N_{r+1}} \subset \frac{N'_2}{N_{r+1}} \subset ....$ of uncountably generated submodules of $\frac{N_r}{N_{r+1}}$. Thus

$$... \subseteq N_{r+2} \subseteq N_{r+1} \subset N'_1 \subset N'_2 \subset ...$$

is a doubly infinite chain of uncountably generated submodules of M. It follows that there exists an integer $k > r$ such that $N_m = N_{m+1}$, for all $m \geq k$. □

The proof of the next lemma is similar to the proof of Lemma 3.2, and it is therefore omitted.

Lemma 3.3. *If M is an u.c.g.-DICC module, then given any infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of uncountably generated submodules of M either $\frac{M_{i+1}}{M_i}$ satisfies the descending chain condition on uncountably generated submodules for all i, or there exists an integer k such that $M_i = M_{i+1}$ for each $i \geq k$.*

The proof of the next lemma is elementary and is omitted.

Lemma 3.4. *An R-module M is an $u.c.g. - DICC$-module if and only if for any uncountably generated submodule A of M either A satisfies the descending chain condition on uncountably generated submodules or $\frac{M}{A}$ satisfies the ascending chain condition on uncountably generated submodules.*

Using Lemma 3.6, we give the next immediate result.

Lemma 3.5. *If M is an $u.c.g. - DICC$ module, then for each uncountably generated submodule X of M, either X or $\frac{M}{X}$ has Krull dimension.*

Proof. Let X be any uncountably generated submodule of M. By Lemma 3.4, either X satisfies the descending chain condition on uncountably generated submodules or $\frac{M}{X}$ satisfies the ascending chain condition on uncountably generated submodules. Hence, by Propositions 2.5, 2.4, and 2.1, either X or $\frac{M}{X}$ has Krull dimension. □

In view of the previous proposition we have the following result.

Proposition 3.6. *If M is an $u.c.g. - DICC$ module, then for each proper countably generated submodule N of M and any uncountably generated submodule X of N either X or $\frac{M}{N}$ has Krull dimension.*

Proof. Let N be a countably generated submodule of M. If X is an uncountably generated submodule of N, then in view of Lemma 3.5, we infer that either X or $\frac{M}{X}$ has Krull dimension. If $\frac{M}{X}$ has Krull dimension, then $\frac{M}{N}$ has Krull dimension; see [12, Lemma 1.1], (note, $\frac{M/X}{N/X} = \frac{M}{N}$). This implies that for each proper countably generated submodule N of M and any uncountably generated submodule X of N either X or $\frac{M}{N}$ has Krull dimension. □

By considering the above proposition and Proposition 2.5, we are now ready to prove the following proposition, which is a crucial step towards proving our main result.

Proposition 3.7. *If M is an u.c.g. $-$ DICC module, then for each proper countably generated submodule N of M either N or $\frac{M}{N}$ has Krull dimension.*

Proof. Suppose that there exists a proper countably generated submodule N' of M such that $\frac{M}{N'}$ does not have Krull dimension. We are to show that N' has Krull dimension. If each proper submodule of N' is countably generated, then N' satisfies the descending chain condition on uncountably generated submodules and in view of Proposition 2.5 we get, N' has Krull dimension and we are through. Otherwise N' has a proper uncountably generated submodule, X' say. In view of Proposition 3.6, we infer that X' has Krull dimension (note, by our assumption $\frac{M}{N'}$ does not have Krull dimension). This implies that any uncountably generated submodule X of N' has Krull dimension. Now, let P be a countably generated submodule of N'. If P is contained in a uncountably generated submodule X of N', then X and therefore P has Krull dimension; see [12, Lemma 1.1]. Otherwise each proper submodule of $\frac{N'}{P}$ is countably generated and this implies that $\frac{N'}{P}$ satisfies descending chain condition on uncountably generated submodules. Therefore $\frac{N'}{P}$ has Krull dimension, see Proposition 2.5. Thus for each submodule X of N', either X or $\frac{N'}{X}$ has Krull dimension; hence, by Proposition 2.3 N' has Krull dimension. □

Next, we present our main result of this paper.

Theorem 3.8. *Let M be a quotient finite dimensional module. If M is an u.c.g. $-$ DICC module, then M has Krull dimension.*

Proof. It suffices to show that M is satisfied in Proposition 2.3. By Lemma 3.5, we infer that for each uncountably generated submodule X of M either X or $\frac{M}{X}$ has Krull dimension. Now, let N be a countably generated submodule of M. By Proposition 3.7, we have either N or $\frac{M}{N}$ has Krull dimension. It follows that for each submodule P of M, either P or $\frac{M}{P}$ has Krull dimension and we are done. □

The next example shows that the converse of the previous theorem is not true in general.

Example 3.9. Let $\mathbb{Z}$ be the ring of integers and B be an ω_1-atomic $\mathbb{Z}$-module; then the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus B \oplus B \oplus B$ has Krull dimension; see [12, Lemma 1.1]. The following chain

$$\mathbb{Z} \oplus B \supset 2\mathbb{Z} \oplus B \supset 4\mathbb{Z} \oplus B \supset \ldots$$

of uncountably generated submodules of the $\mathbb{Z}$-module $\mathbb{Z} \oplus B$ shows that it does not satisfy the descending chain condition on uncountably generated submodules. Since B is not Noetherian, hence there exits the following chain $B_1 \subset B_2 \subset B_3 \subset \ldots$ of submodules of B. Thus the following chain

$$B_1 \oplus B \subset B_2 \oplus B \subset B_3 \oplus B \subset \ldots$$

of uncountably generated submodules of the $\mathbb{Z}$-module $B \oplus B$ shows that it does not satisfy the ascending chain condition on uncountably generated submodules. Since $\frac{M}{\mathbb{Z} \oplus B} \simeq B \oplus B$, we infer that M is not an $u.c.g.$-DICC module by Lemma 3.4.

It is clear that if an R-module M satisfies the ascending or the descending chain condition on uncountably generated submodules, then it is $u.c.g. - DICC$. Also, it is evident that any $DICC$ module is $u.c.g. - DICC$, but the converse is not true in general. For example, it is clear that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ satisfies the ascending chain condition on uncountably generated submodules, and thus it is an $u.c.g. - DICC$-module. Clearly, $\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ is not $DICC$; see the comment that follows [13, Definition 1.1].

The following result is clear and its proof is omitted.

Corollary 3.10. *Let M be an R-module and N be a proper submodule of M. If M is an $u.c.g. - DICC$ module, then so are N and $\frac{M}{N}$.*

We recall that a composition series for a module M is a chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each of the factors $\frac{M_i}{M_{i-1}}$ is a simple module. A module of finite length is any module that has a composition series. Moreover, it is well known that a module M has finite length if and only if M is both Noetherian and Artinian.

Note the following result. The proof is standard but we include it for completeness.

Corollary 3.11. *Let M be an R-module and let N be of finite length submodule of M. If $\frac{M}{N}$ is $u.c.g. - DICC$, then so is M.*

Proof. Let $\ldots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ be a double infinite chain of uncountably generated submodules of M. Then $\ldots \subseteq \frac{M_{-2}+N}{N} \subseteq \frac{M_{-1}+N}{N} \subseteq \frac{M_0+N}{N} \subseteq$

$\frac{M_1+N}{N} \subseteq \frac{M_2+N}{N} \subseteq ...$ is a double infinite chain of uncountably generated submodules of $\frac{M}{N}$. Thus, there exists an integer number i such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \geq i$, or there exists an integer number i_1 such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \leq i_1$. Without less of generality, we consider that there exists an integer number i such that $\frac{M_k+N}{N} = \frac{M_{k+1}+N}{N}$ for each $k \geq i$. Thus, we get $M_k + N = M_{k+1} + N$ for each $i \geq k$. However, it is clear that $... \subseteq M_{-2} \cap N \subseteq M_{-1} \cap N \subseteq M_0 \cap N \subseteq M_1 \cap N \subseteq ...$ is a double infinite chain of submodules of N. Since N has finite length, it follows that there exists an integer number i_2 such that $M_k \cap N = M_{k+1} \cap N$ for each $k \geq i_2$. Put $n = \max\{i_1, i_2\}$. Thus $M_k + N = M_n + N$ and $M_k \cap N = M_n \cap N$ for each $k \geq n$. For each $k \geq n$, we conclude that $M_k = M_k \cap (M_k + N) = M_k \cap (M_n + N) = M_n + (M_k \cap N) = M_n + (M_n \cap N) = M_n$ and we are through. □

Finally, we investigate when atomic modules are *u.c.g.*-DICC modules.

Proposition 3.12. *Let $\alpha > \omega_1$ be an ordinal number. An α-atomic module M is $u.c.g.-DICC$ if and only if M satisfies the descending chain condition on uncountably generated submodules.*

Proof. The sufficiency is obvious. Conversely, since M is α-atomic, we infer that for each proper submodule N of M, n-dim $\frac{M}{N} = \alpha$. By Proposition 2.4, $\frac{M}{N}$ does not satisfy the ascending chain condition on uncountably generated submodules. Now, by Lemma 3.4, N satisfies the descending chain condition on uncountably generated submodules and so does M, see[9, Lemma 4.4]. □

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