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NON-PARALLEL GRAPH OF SUBMODULES OF A MODULE

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ABSTRACT. Let R be a ring with identity and M be a unitary left R-module. A non-parallel submodules graph of M, denoted by $G_{\nparallel}(M)$, is an undirected simple graph whose vertices are in one-one correspondence with all non-zero proper submodules of M and two distinct vertices are adjacent if and only if they are not parallel to each other. In this paper, we investigate the interplay between some module-theoretic properties of M and graph-theoretic properties of $G_{\nparallel}(M)$. It is shown that if $G_{\nparallel}(M)$ is connected, then $\operatorname{diam}(G_{\nparallel}(M)) \leq 3$ and if $G_{\nparallel}(M)$ is not connected, then $G_{\nparallel}(M)$ is a null graph. It is proved that $G_{\nparallel}(M)$ is null if and only if M contains a unique simple submodule. In particular, M is strongly semisimple R-module if and only if $G_{\nparallel}(M)$ is a complete graph, and from this, it follows that if $G_{\nparallel}(M)$ is complete, then every R-module with finite Goldie dimension is Artinian and Noetherian. In addition, $G_{\nparallel}(M)$ is a finite star graph if and only if $M \cong \mathbb{Z}_{pq}$, for some distinct prime numbers p and q.

1. Introduction and preliminaries

The investigation of connections between the algebraic structures' theoretic properties and the graph-theoretic properties has been studied by several authors. In 1964, Bosak introduced the concept of the graph of semigroups, see [5]. Inspired by his work, in 1969, Csakany and Pollak, studied the graph of subgroups of a finite group, in [6]. Fundamental papers devoted to graphs assigned to a ring have appeared, see [3, 2]. In this article, we associate a graph to a module over an arbitrary ring (not necessarily commutative). Our

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main goal is to study the connection between the algebraic properties of a module and the graph-theoretic properties of the graph associated with it. Two modules A and B are orthogonal, written as $A \perp B$, if they do not have non-zero isomorphic submodules. Modules A and B are parallel, denoted by $A \parallel B$, if there does not exist non-zero submodule of A which is orthogonal to B and also there does not exist non-zero submodule of B which is orthogonal to A. A module M is called atomic if all of its non-zero submodules are parallel to each other and so they are parallel to M itself. For more details and some basic facts about atomic modules, the reader is referred to [7, 8]. We should remind the reader that these atomic modules are different from those defined in [9]. In this paper, we introduce and study the concept of non-parallel graph of submodules of an R-module M, denoted by $G_{\nparallel}(M)$, that is, the undirected simple graph with the vertices set $\mathrm{V}(G_{\nparallel}(M))$ whose vertices are in one-one correspondence with all non-zero proper submodules of M and two distinct vertices A and B are adjacent if and only if $A \nparallel B$. Let G be an undirected graph. We say that the graph G is connected, if there is a path between any two distinct vertices. By a null graph, we mean a graph with no edges. A x, y-path is a path with starting vertex x and ending vertex y. For distinct vertices x and y, let $d(x, y)$ be the least length of an x, y-path. If G has non such a path, then $d(x, y) = \infty$. The diameter of G, is the supremum of the set $\{d(x, y) \mid \ x \text{ and } y \text{ are vertices of } G\}$. A cycle of length n in G is a path of the form $x_1 - x_2 - x_3 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of the smallest cycle in G, provided G contains a cycle, otherwise $\mathrm{gr}(G) = \infty$. A *complete graph* is a graph in which every pair of distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. By a complete subgraph, we mean a subgraph which is complete as a graph. A *bipartite graph* (or bigraph) is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 (that is, V_1 and V_2 are each independent sets) such that every edge connects a vertex in V_1 to one in V_2. Assume that $K_{m,n}$ denoted the complete bipartite graph on two nonempty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (here m and n may be infinite cardinal number). In particular, $K_{1,n}$ is called a *star graph*, that is, a tree consisting of one vertex adjacent to all the others. In graph theory, an independent set or stable set is a set of vertices in a graph, no two of which are adjacent. Let us give a brief outline of this article. After reviewing some necessary preliminaries, in Section 2, we show that if $G_{\nparallel}(M)$ is connected, then $\mathrm{diam}(G_{\nparallel}(M)) \leq 3$ and if $G_{\nparallel}(M)$ is not connected, then $G_{\nparallel}(M)$ is a null graph. It

is proved that $G_{\nparallel}(M)$ is null if and only if M contains a unique simple submodule. We recall that the Goldie dimension of an R-module M, denoted by G-dim$\,(M)$, which is the supremum of all cardinals k such that M contains a direct sum of k non-zero submodules. In particular, it is shown that M is strongly semisimple R-module if and only if $G_{\nparallel}(M)$ is a complete graph and as a result, if $G_{\nparallel}(M)$ is complete, then every R-module with finite Goldie dimension is Artinian and Noetherian. In Section 3, we provide some examples of non-parallel graphs of cyclic finite Abelian groups. It is proved that $G_{\nparallel}(M)$ is a finite star graph if and only if $M \cong \mathbb{Z}_{pq}$, for some distinct prime numbers p and q.
Throughout this article, all rings are associative with $1 \neq 0$ and all modules are unital left modules. The notation $A \subseteq_e M$ means A is an essential submodule of M. A module M is said to be uniform if every non-zero submodule of M is essential. A non-zero R-module M is said to be simple if it has no non-trivial submodule. The socle of an R-module M, written $\mathrm{Soc}(M)$, is the sum of all simple submodules of M. For an R-module M, the length of M, is denoted by $l_R(M)$.

2. Connectivity, Diameter and Girth of $G_{\nparallel}(M)$

In this section, we characterize all modules for which the non-parallel graph of submodules, i.e., $G_{\nparallel}(M)$, is connected. Also, the diameter and the girth of $G_{\nparallel}(M)$ are determined. Moreover, we study some modules whose non-parallel graphs are complete.

We need the following result, see also [12].

Lemma 2.1. *Let M be an R-module and A, B, C be submodules of M. Then the following statements hold.*

(1) *If $A \subseteq_e M$, then $A \parallel M$.*
(2) *If $B \cong C$ and $A \perp B$, then $A \perp C$.*
(3) *If $A \parallel B$ and $B \parallel C$, then $A \parallel C$.*
(4) *If $A \parallel B$ such that $C \perp A$, then $C \perp B$.*
(5) *If $A \parallel B$ such that $B \cong C$, then $A \parallel C$.*
(6) *If $C \subseteq B \subseteq A$ such that $C \parallel A$, then $B \parallel A$.*

Theorem 2.2. *Let M be an R-module. If $G_{\nparallel}(M)$ is connected, then* $\mathrm{diam}(G_{\nparallel}(M)) \leq 3$.

Proof. Let N and K be two non-trivial distinct submodules of M. If $N \nparallel K$, then $d(N, K) = 1$. Suppose that $N \parallel K$ and so $N \not\perp K$. In this case, there exists $0 \neq N_1 \subseteq N$

and $0 \neq K_1 \subseteq K$ such that $N_1 \cong K_1$. Since $G_{\nparallel}(M)$ is connected, so there exist non-zero submodules N' and K' of M such that N is adjacent to N' and K is adjacent to K'. Now, the following cases may happen.

Case 1. If $N' = K'$, then $N - N' - K$ is a path of length 2, that is, $d(N, K) = 2$.

Case 2. If N' is adjacent to K', then $N - N' - K' - K$ is a path of length 3, that is, $d(N, K) = 3$.

Case 3. If N' is not adjacent to K', then $N' \parallel K'$ and so $N' \not\perp K'$. Thus, there exist $0 \neq N'' \subseteq N'$ and $0 \neq K'' \subseteq K'$ such that $N'' \cong K''$. Now, since $K \nparallel K'$ two cases may happen:

Case (i): There exists $0 \neq K_2 \subseteq K$ such that $K_2 \perp K'$. Since $K'' \subseteq K'$, whence $K_2 \perp K''$ and by Lemma 2.1(2), $K_2 \perp N''$, that is, N' and K contain orthogonal submodules. Hence, $N' \nparallel K$ and so $N - N' - K$ is a path of length 2, that is, $d(N, K) = 2$.

Case (ii): There exists $0 \neq K'_1 \subseteq K'$ such that, $K'_1 \perp K$. Since $K_1 \subseteq K$, so $K'_1 \perp K_1$. But $K_1 \cong N_1$ and so $K'_1 \perp N_1$ hence, N and K' contain orthogonal submodules. Thus, $N \nparallel K'$ and $N - K' - K$ is a path of length 2, that is, $d(N, K) = 2$. □

Proposition 2.3. *Let M be an R-module. If $G_{\nparallel}(M)$ is not connected, then $G_{\nparallel}(M)$ is a null graph. Moreover, M is an atomic module.*

Proof. Assume that $G_{\nparallel}(M)$ is not connected and C_1 and C_2 are two components of $G_{\nparallel}(M)$. Let N and K be two submodules of M such that $N \in C_1$ and $K \in C_2$. Since N and K are not adjacent thus, $N \parallel K$. It suffices to show that, M is an atomic module. On the contrary, there exist $A, B \subseteq M$ such that $A \nparallel B$. Now, we put $A, B \in C_1$ thus, $A \parallel K$ and $B \parallel K$, by Lemma 2.1(3), we get $A \parallel B$, the contradiction required. □

Theorem 2.4. *Let M be an R-module. If $G_{\nparallel}(M)$ contains a cycle, then* $\mathrm{gr}(G_{\nparallel}(M)) = 3$.

Proof. Let A and B be two distinct non-trivial submodules of M with $A \nparallel B$. Then two cases may happen:

Case 1. There exists $0 \neq A_1 \subseteq A$ such that $A_1 \perp B$. It is easy to see that $A \nparallel A + B$ and $B \nparallel A + B$. Thus (A , A+B, B) is a cycle of length 3 in $G_{\nparallel}(M)$.

Case 2. There exists $0 \neq B_1 \subseteq B$ such that $A \perp B_1$. By similar way of case 1, we are done. □

Remark 2.5. We recall that an R-module M is said to be atomic if every pair of non-zero submodules of M are parallel. Clearly, $\emptyset \neq G_{\nparallel}(M)$ is a null graph if and only if M is an atomic module. For instance, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, and $\mathbb{Z}_{p^\infty}$, where p is a prime number, as $\mathbb{Z}$-module, are atomic and hence, their non-parallel graphs are null. But semisimple modules, which they have at least two non-isomorphic simple submodules, can not have null non-parallel graph.

Proposition 2.6. *Let M be an Artinian R-module. Then $G_{\nparallel}(M)$ is null if and only if M contains a unique simple submodule.*

Proof. Suppose that M contains a unique simple submodule, say S. Since M is Artinian thus, for all submodules A, B of M, $S \subseteq A \cap B$, and then $A \parallel B$, that is, $G_{\nparallel}(M)$ is a null graph. Conversely, if $G_{\nparallel}(M)$ is null, then M is an atomic module. But M is Artinian, thus it has a simple submodule. We assume that S_1 and S_2 are two simple submodule of M thus, $S_1 \parallel S_2$ and so $S_1 = S_2$. Therefore, M contains a unique simple submodule. □

The following result is immediate.

Corollary 2.7. *Let M be an Artinian R-module. Then M is atomic if and only if M contains a unique simple submodule.*

Inasmuch as empty graph is trivially a complete graph, we shall focus on this question that under which conditions $G_{\nparallel}(M)$ is a complete graph. In the following theorem, we study some modules whose non-parallel graph of submodules are complete. A module M is called strongly semisimple, if M be semisimple and has no isomorphic simple submodules.

Theorem 2.8. *Let M be an R-module. Then M is strongly semisimple if and only if $G_{\nparallel}(M)$ is a complete graph.*

Proof. Suppose that M is a strongly semisimple module, so $M = \mathrm{Soc}(M) = \oplus_{i \in I} S_i$ such that $S_i \ncong S_j$ for any $i \neq j$. Assume that $K \neq N$ are non-zero proper submodules of M, so there exist $S_1 \subseteq M \backslash N$ and $S_2 \subseteq M \backslash K$. But $S_1 \cap N \subset S_1$ and $S_2 \cap K \subset S_2$ thus, $S_1 \cap N = 0$ and $S_2 \cap K = 0$. Now, we claim that $S_1 \perp N$ and $S_2 \perp K$. Suppose that $S_1 \not\perp N$, there exists $0 \neq N_1 \subseteq N$ such that $N_1 \cong S_1$. Therefore, $S_1 = S_1 \cap S_1 \cong N_1 \cap S_1 \subseteq N \cap S_1 = 0$ and then $S_1 = 0$, which is a contradiction. A similar argument shows that $S_2 \perp K$. Thus, $S_1 \nparallel N$ and $S_2 \nparallel K$. But N and K are semisimple, so there exist non-isomorphic simple submodules T and T' of M such that $T \subseteq N$ and $T' \subseteq K$ (note, $N \neq K$). Thus, N, K contain orthogonal submodules and so $N \nparallel K$. Therefore, the graph $G_{\nparallel}(M)$ is complete, that is, M does not have parallel submodules. Conversely, suppose that $G_{\nparallel}(M)$ is a complete graph. We show that M is semisimple. To see this, it suffices to show that, M has no essential submodule. If $A \subseteq_e M$, then for any $0 \neq B \subseteq M$, $A \cap B \neq 0$ and so $A \not\perp B$, that is, $A \parallel B$ and the contradiction required. Moreover, if $M = \mathrm{Soc}(M) = \oplus_{i \in I} S_i$ and there exists $S_i \cong S_j$ for some $i \neq j$, then $S_i \parallel S_j$, which is a contradiction. Hence, M is strongly semisimple. □

We note that the strongly condition in Theorem 2.8, is required. Because, if M is a semisimple module which contains isomorphic simple submodules S_1 and S_2, then $S_1 \parallel S_2$. This is contradictory with the completeness of the graph.

Corollary 2.9. *Let M be an R-module with finite Goldie dimension. If $G_{\nparallel}(M)$ is a complete graph, then M is both Artinian and Noetherian.*

Proof. Since G-dim$\,(M) < \infty$, so for any chain $M_0 \subsetneq M_1 \subsetneq ...$ of submodules of M, there exists n such that for any $k \geq n$, $M_n \subseteq_e M_k$. Hence, by Lemma 2.1(1), $M_n \parallel M_k$, but $G_{\nparallel}(M)$ is complete and so $M_n = M_k$. This shows that M is Noetherian. Applying the same argument to any chain $M_0 \supsetneq M_1 \supsetneq ...$ of submodules of M, we get M is Artinian. □

The next example shows that, the converse of Corollary 2.9, is not true in general.

Example 2.10. Let $M = \mathbb{Z}_{18}$ be a $\mathbb{Z}$-module . It is easy to see that M is Artinian and Noetherian and so we infer that G-dim$\,(M) < \infty$. But the following figure shows that $G_{\nparallel}(M)$ is not complete.

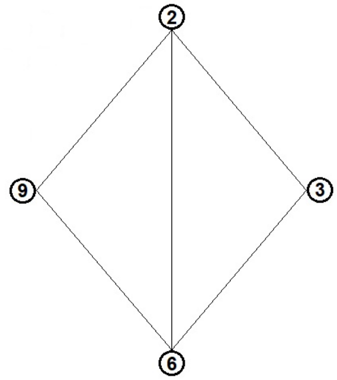

FIGURE 1. $\mathbb{Z}_{18}$

Definition 2.11. Let M be an R-module. The Krull dimension of M, denoted by $k\text{-dim}\,(M)$ is defined by transfinite recursion as follows: If $M = 0$, $k\text{-dim}\,(M) = -1$. If α is an ordinal number and $k\text{-dim}\,(M) \not< \alpha$, then $k\text{-dim}\,(M) = \alpha$ provided there is no infinite descending chain of submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ such that for each $i = 1, 2, \ldots$, $k\text{-dim}\,(\frac{M_{i-1}}{M_i}) \not< \alpha$. In otherwise $k\text{-dim}\,(M) = \alpha$, if $k\text{-dim}\,(M) \not< \alpha$ and for each chain of submodules to M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ there exists an integer t, such that for each $i \geq t$, $k\text{-dim}\,(\frac{M_{i-1}}{M_i}) < \alpha$. A ring R has Krull dimension, if as an R-module has Krull dimension. It is possible that there is no ordinal α such that $k\text{-dim}\,(M) = \alpha$, in this case we say M has no Krull dimension.

It is well known that, every module with Krull dimension has finite Goldie dimension.

Proposition 2.12. *Let M be an R-module with Krull dimension. If $G_{\nparallel}(M)$ is complete, then M is both Artinian and Noetherian.*

Since $\mathbb{Z}$ is not Artinian and, $\mathbb{Z}_{p^\infty}$ is not Noetherian, as $\mathbb{Z}$-module, hence $\mathbb{Z}$-module $Z \oplus Z_{p^\infty}$ is neither Artinian nor Noetherian, but it has finite Goldi dimension (note, every module with Krull dimension has finite Goldie dimension). Therefore, we have the following result.

Corollary 2.13. *$G_{\nparallel}(\mathbb{Z})$, $G_{\nparallel}(\mathbb{Z}_{p^\infty})$ and $G_{\nparallel}(\mathbb{Z} \oplus \mathbb{Z}_{p^\infty})$, as $\mathbb{Z}$-module, are not complete.*

Recall that, a path graph with 2 vertices is denoted by P_2. We note that if $G_{\nparallel}(M) \cong P_2$, then $\mathrm{V}(G_{\nparallel}(M)) = \{A, B\}$ such that $A \nparallel B$. It is easy to see that A and B are non-isomorphism simple submodules of M thus, $A \perp B$ and then $A \cap B = 0$. Since $A + B = M$, whence $M = A \oplus B$ thus, $l_R(M) = 2$, for example $\mathbb{Z}$-module $\mathbb{Z}_6$, see the following figure.

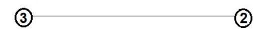

FIGURE 2. $\mathbb{Z}_6$

3. Non-Parallel Graph For Abelian Groups

Inasmuch as Abelian groups are precisely $\mathbb{Z}$-modules. For the sake of the reader we provide some examples of non-parallel graphs of cyclic finite Abelian groups, i.e., $\mathbb{Z}_n$, where $n \in \mathbb{N}$. While straightforward, we need the following lemma to prove the next results.

Lemma 3.1. *let $A =< m >$ and $B =< k >$ are subgroups of $\mathbb{Z}_n$, then the following hold:*

(1) *$A \perp B$ if and only if $(O(m), O(k)) = 1$.*
(2) *$A \not\parallel B$ if and only if for any $m'|m$ and $k'|k$, $(O(m'), O(k')) = 1$.*

Proof. (1) See [10, Lemma 5.1].
(2) Assume that $m'|m$, $k'|k$ and $(O(m'), O(k')) = 1$, by part of (1), $< m' >\perp< k' >$, i.e., A and B have orthogonal submodules and thus, $A \not\parallel B$.
Conversely, let $A \not\parallel B$ so, there exists non-zero submodule $A' =< m' >$ of A such that, either $A' \perp B$ or there exists non-zero submodule $B' =< k' >$ of B such that $A \perp B'$. By part of (1), $(O(m'), O(B)) = (O(m'), O(k)) = 1$ and we infer that $(O(m'), O(k')) = 1$, for any $k'|k$. Similarly, $(O(k'), O(A)) = (O(k'), O(m)) = 1$ and hence, $(O(k'), O(m')) = 1$ for any $m'|m$. □

In the next theorem, we characterize finite Abelian groups for which their non-parallel graphs are star graphs.

Theorem 3.2. *Let M be a finite Abelian group. Then $G_{\not\parallel}(M)$ is a finite star graph if and only if $M \cong \mathbb{Z}_{pq}$, for some distinct prime numbers p and q.*

Proof. Since M is a finite Abelian group thus, there exist prime numbers $p_1, p_2, ..., p_m$ and positive integers $\alpha_1, \alpha_2, ..., \alpha_m$, such that $M \cong \mathbb{Z}_{p_1}^{\alpha_1} \oplus ... \oplus \mathbb{Z}_{p_m}^{\alpha_m}$. It suffices to show that

$m = 2$ and $\alpha_1 = \alpha_2 = 1$. Hence, the proof will be divided into two steps.

Step 1. We show that $m = 2$. Suppose, by way of contradiction, that $m \geq 3$. Put $N_1 = \mathbb{Z}_{p_1}^{\alpha_1} \oplus \{0\} \oplus ... \oplus \{0\}$, $N_2 = \{0\} \oplus \mathbb{Z}_{p_2}^{\alpha_2} \oplus ... \oplus \{0\}$ and $N_3 = \{0\} \oplus \{0\} \oplus \mathbb{Z}_{p_3}^{\alpha_3} \oplus ... \oplus \{0\}$. It is clear that N_1, N_2 and N_3 are three distinct subgroups of M, which are mutually adjacent in $G_{\nparallel}(M)$. It is a contradiction because $G_{\nparallel}(M)$ is a star graph.

Step 2. We show that $\alpha_1 = \alpha_2 = 1$. Assume that $\alpha_1 \geq 2$ and $\alpha_1 \geq 2$. It is easy to see that $G_{\nparallel}(\mathbb{Z}_{p_1}^{\alpha_1} \oplus \mathbb{Z}_{p_2}^{\alpha_2})$ is not a star graph. Hence, $M \cong \mathbb{Z}_{pq}$. $\square$

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