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A FINITE DIFFERENCE METHOD FOR SMOOTH SOLUTION OF A SYSTEM OF LINEAR VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper, we propose a numerical method for the smooth solution of a system of linear Volterra integral equations. This method is a generalization of the finite difference method proposed in [11] and [12] for scalar linear Volterra integral equations. Error analysis of this method is presented via asymptotic expansion of the absolute error, and verification of accuracy is examined by two illustrative test problems.

1. Introduction

Extracting the formulation of a large number of physical models in terms of integral equations (IEs) is a classical theoretical methodology wildly utilized in literature [5, 6, 10, 8, 14, 15]. Linear and nonlinear Volterra and Fredholm integral equations have been studied extensively by many authors using various methods and techniques [2, 3, 7, 13]. The present study is concerned with the numerical solution of a system of linear weakly singular Volterra integral equations (VIEs) in conjunction with extrapolation procedures. Let $T_f > 0$ and $D = \{(t, \tau) : 0 \leq \tau \leq t \leq T_f\}$. Consider the Banach spaces of the vector and matrix valued functions $V = C([0, T_f]; \mathbb{R}^N)$ and $W = C(D; M_{N\times N}(\mathbb{R}))$ equipped with the following norms

$$\|\mathbf{f}\|_V = \sup_{0\leq t\leq T_f} \|\mathbf{f}(t)\|_2, \quad \mathbf{f} \in V,$$

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and

$$\|\mathbf{K}\|_W = \sup_{(t,\tau)\in D} \|\mathbf{K}(t,\tau)\|_2, \quad \mathbf{K} \in W,$$

where $\|.\|_2$ is the standard vector, matrix 2- norms

$$\|\mathbf{v}\|_2 = (\mathbf{v}^T.\mathbf{v})^{\frac{1}{2}}, \mathbf{v} \in \mathbb{R}^N; \quad \|\mathbf{K}\|_2 = \sup_{0\neq\mathbf{v}\in\mathbb{R}^N} \frac{\|\mathbf{K}\mathbf{v}\|_2}{\|\mathbf{v}\|_2},$$

and T is the standard transpose operator. In this paper, for the sake of clarity and brevity, all matrices and matrix valued functions are denoted in upper case, all vectors are denoted in lower case bold and scalars are denoted by plain lower case. Finally, the superscripts and subscripts are mostly used for component and function values at nodal points or as index respectively. This study provides an extension of finite difference method proposed in [12] and [11] to solve the system of linear weakly singular VIEs of second kind as follows

$$\mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{K}(t,\tau)\mathbf{\Phi}(t,\tau)\mathbf{u}(\tau)d\tau, \tag{1.1}$$

where the functions $\mathbf{f}(t) = [f^1(t), f^2(t), ..., f^N(t)]^T \in V$, and $\mathbf{K}(t,\tau) = [k^{pq}(t,\tau)] \in W$, are known and $\mathbf{u}(t) = [u^1(t), u^2(t), ..., u^N(t)]^T$ is unknown. In the context of the weakly singular integral equations, we assume

$$\mathbf{\Phi}(t,\tau) = [\varphi^{pq}(t,\tau)] \in M_{N\times N}(\mathbb{R}),$$

where φ^{pq} is of either the form

Case I.: $\varphi^{pq}(t,\tau) = (t-\tau)^{-\nu_{pq}}, \quad \nu_{pq} \in \mathbb{R}, \quad \nu_{pq} < 0$ (Regular type),

Case II.: $\varphi^{pq}(t,\tau) = (t-\tau)^{-\nu_{pq}}, \quad \nu_{pq} \in \mathbb{R}, \quad 0 < \nu_{pq} < 1$ (Abel type),

Case III.: $\varphi^{pq}(t,\tau) = log\gamma_{pq}(t-\tau), \quad \gamma_{pq} \in \mathbb{R}, \quad \gamma_{pq} > 0$ (Logarithmic type).

We can easily show that the system (1.1) can be transformed to the following system of integral equations

$$\mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{K}(t,\tau)\hat{\mathbf{\Phi}}(t,\tau)(t-\tau)^{-\nu}\mathbf{u}(\tau)d\tau, \ 0 < v < 1, \tag{1.2}$$

where $\nu = \max\ \nu_{pq}$ and for all $t \neq \tau$

$$\hat{\varphi}^{pq}(t,\tau) = (t-\tau)^{\nu-\nu_{pq}}, \ 0 < \nu_{pq} < 1,$$

or

$$\hat{\varphi}^{pq}(t,\tau) = (t-\tau)^{\nu} log\gamma_{pq}(t-\tau).$$

It is clear that in I,II and III, the matrix valued function $\hat{\mathbf{\Phi}}$ is continuous on D or more briefly $\hat{\mathbf{\Phi}} \in W$. Similar to the scalar equations, the solvability results for the system (1.1) can be achieved via the system (1.2). It is worth to note that the existence and uniqueness of solution of the system (1.2) may be proved by a formal generalization of the scalar case [2, 11].

It is shown that for given smooth vector and matrix functions $\mathbf{f}$ and $\mathbf{K}$, the unique solution of the equation (1.1) may have weak singularity at the end point of the interval $[0, T_f)$ [3, 9]. However, there are integral equations (possibly originating from an initial boundary value problem or systems of initial value problems) wherein the smoothness of their solution is known on $[0, T_f]$.

In [12] and [11], a finite difference method is proposed for the numerical solution of (1.1) in the scalar case, where the corresponding system for the discretized problem leads to a lower triangular matrix. In this paper, we consider a generalization of this method for a system of weakly singular Volterra integral equations and show that it leads to a block in the lower triangular system. A comparison between this method and the standard product integration method shows the advantages and drawbacks of the finite difference method.

This paper is organized as follows: In section 2, the numerical scheme is described. In section 3, the solvability and accuracy of the discretized problem obtained using the proposed finite difference method are analyzed. Section 4 contains two numerical examples to show the robustness of the numerical method for solving regular and weakly singular VIEs.

2. **Numerical Scheme**

For a given positive integer n, let $0 = t_0 < t_1 < ... < t_n = T_f$ be a partition of $[0, T_f]$, $t_i = ih, h = T_f/n$, $\mathbf{f}_i = \mathbf{f}(t_i)$, $\mathbf{u}_i = \mathbf{u}(t_i)$, and $\mathbf{K}_{ij} = \mathbf{K}(t_i, t_j)$, for $j = 0, 1, ..., i$ and $i = 0, 1, ..., n$.

To discretize the integral equation, we collocate (1.1) at the grid points $t = t_i$ as follows

$$\mathbf{u}(t_i) = \mathbf{f}(t_i) + \int_0^{t_i} \mathbf{K}(t_i, \tau)\mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau, \tag{2.1}$$

or

$$\mathbf{u}(t_i) = \mathbf{f}(t_i) + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \mathbf{K}(t_i, \tau)\mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau. \tag{2.2}$$

For simplicity, we derive the numerical scheme in two steps.

- STEP 1. On each subinterval $[t_{j-1}, t_j]$, we approximate $\mathbf{K}(t_i, \tau)$ by $\mathbf{K}(t_i, t_j)$ and write

$$\mathbf{K}(t_i, \tau) = \mathbf{K}_{ij} + \Lambda_{ij}^{(11)}, \quad \tau \in [t_{j-1}, t_j], \tag{2.3}$$

where $\mathbf{K}_{ij}, \Lambda_{ij}^{(11)} \in M_{N\times N}(\mathbb{R})$ and $\Lambda_{ij}^{(11)}$ denotes the truncation error of approximation of $\mathbf{K}$ see [13]. Therefore, for $i = 1, ..., n$, we have

$$\begin{aligned} \mathbf{u}_i &= \mathbf{f}_i + \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \left[\mathbf{K}_{ij} + \Lambda_{ij}^{(11)}\right] \mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau \\ &= \mathbf{f}_i + \sum_{j=1}^{i} \mathbf{K}_{ij} \int_{t_{j-1}}^{t_j} \mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau + \Lambda_i^{(12)}, \end{aligned} \tag{2.4}$$

where $\Lambda_i^{(12)} \in \mathbb{R}^N$ is the truncation error of approximation of $\mathbf{u}$ on $[0, t_i]$. For notation simplicity let

$$\mathbf{u}_i = \mathbf{f}_i + \sum_{j=1}^{i} \mathbf{K}_{ij}\Omega_{ij} + \Lambda_i^{(12)}, \tag{2.5}$$

where

$$\Omega_{ij} = \int_{t_{j-1}}^{t_j} \mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau, \quad \Omega_{ij} \in M_{N\times N}(\mathbb{R}). \tag{2.6}$$

- STEP 2. First by using the integral by parts scheme(see [17]) the following identity can be derived

$$\int_{t_{j-1}}^{t_j} \mathbf{\Phi}(t_i, \tau)\mathbf{u}(\tau)d\tau = \mathbf{Q}_{ij}(t_j)\mathbf{u}(t_j) - \int_{t_{j-1}}^{t_j} \mathbf{Q}_{ij}(s)\mathbf{u}'(s)ds, \tag{2.7}$$

where

$$\mathbf{Q}_{ij}(t) = \int_{t_{j-1}}^{t} \mathbf{\Phi}(t_i, \tau)d\tau.$$

Next, we apply the finite difference formula to the integral operator on the right hand side of (2.7). We can write

$$\Omega_{ij} = \mathbf{Q}_{ij}(t_j)\mathbf{u}_j - \left(\frac{\mathbf{u}_j - \mathbf{u}_{j-1}}{h} \int_{t_{j-1}}^{t_j} \mathbf{Q}_{ij}(s)ds + \Lambda_{ij}^{(21)} \right), \tag{2.8}$$

where $\Lambda_{ij}^{(21)}$ denotes the local truncation error on the subinterval $[t_{j-1}, t_j]$.

Since Ω_{ij} in (2.8) is a linear combinations of $\mathbf{u}_j$ and $\mathbf{u}_{j-1}$, we can rearrange and write the equation(2.8) as

$$\Omega_{ij} = \mathbf{A}_{ij}\mathbf{u}_j + \mathbf{B}_{ij}\mathbf{u}_{j-1} - \Lambda_{ij}^{(21)}, \quad \mathbf{A}_{ij}, \quad \mathbf{B}_{ij}, \quad \Lambda_{ij} \in M_{N\times N}(\mathbb{R}), \tag{2.9}$$

where

$$\mathbf{A}_{ij} = \mathbf{Q}_{ij}(t_j) - \frac{1}{h}\int_{t_{j-1}}^{t_j} \mathbf{Q}_{ij}(\tau)d\tau, \quad \mathbf{B}_{ij} = \frac{1}{h}\int_{t_{j-1}}^{t_j} \mathbf{Q}_{ij}(\tau)d\tau. \tag{2.10}$$

Substituting Ω_{ij} from (2.9) in (2.5) gives

$$\begin{aligned} \mathbf{u}_i &= \mathbf{f}_i + \sum_{j=1}^{i} \mathbf{K}_{ij} \left(\mathbf{A}_{ij}\mathbf{u}_j + \mathbf{B}_{ij}\mathbf{u}_{j-1} - \Lambda_{ij}^{(21)} \right) + \Lambda_i^{(12)} \\ &= \mathbf{f}_i + \sum_{j=1}^{i} \mathbf{K}_{ij} \left(\mathbf{A}_{ij}\mathbf{u}_j + \mathbf{B}_{ij}\mathbf{u}_{j-1} \right) + \Lambda_i^{(22)}, \end{aligned} \tag{2.11}$$

where $\Lambda_i^{(22)} \in \mathbb{R}^N$ is defined in terms of the truncation errors $\Lambda_i^{(12)}$ and $\Lambda_{ij}^{(21)}$. Equation (2.11) together with the initial condition $\mathbf{u}_0 = \mathbf{f}_0$ characterizes the following finite difference method.

Remark 2.1. The recursive process

$$\begin{cases} \mathbf{u}_i^h = \mathbf{f}_i + \sum_{j=1}^{i} \mathbf{K}_{ij}(\mathbf{A}_{ij}\mathbf{u}_j^h + \mathbf{B}_{ij}\mathbf{u}_{j-1}^h), \quad i = 1, \ldots, n, \\ \mathbf{u}_0^h = \mathbf{f}_0, \end{cases} \tag{2.12}$$

generates an approximate solution for the VIE (1.1), where $\mathbf{u}_i^h = [u_i^{1h}, u_i^{2h}, ..., u_i^{Nh}]^T$ is an approximation to $\mathbf{u}_i = [u_i^1, u_i^2, ..., u_i^N]^T$, provided that $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ satisfy the (2.10).

Now if we introduce $\mathbf{U} = [u_0, u_1, ..., u_n]^T$, then $\mathbf{U}^h = [u_0^h, u_1^h, ..., u_n^h]^T$ may be considered as an approximation for $\mathbf{U}$.

Remark 2.2. Determining the unknown vector $\mathbf{u}_i^h$ at each step of the numerical scheme (2.12) involves a solution of $N \times N$ linear system. The coefficient matrix of this system is $I_N - \mathbf{K}_{ii}\mathbf{A}_{ii}$ where I_N is the identity matrix in $M_{N\times N}(\mathbb{R})$.

Remark 2.3. The numerical scheme (2.12), is equivalent to the linear system

$$M^h \mathbf{U}^h = \mathbf{F}^h, \tag{2.13}$$

where $\mathbf{U}^h \in \mathbb{R}^{N(n+1)}$ is unknown, M^h is a block lower triangular matrix and $\mathbf{F}^h = [f_0, f_1, ..., f_N]^T \in \mathbb{R}^{N(n+1)}$.

Remark 2.4. Using back substitution algorithm, the linear system (2.13) can be solved without storing the coefficient matrix completely. This is one of the advantages of the finite difference method that reduces computer storage.

Remark 2.5. The linear system (2.13) can also be solved using a parallel back substitution algorithms. This property (blocked lower triangular) makes implementation of the finite difference method faster than the other methods.

Remark 2.6. The shape of assembled blocked lower triangular matrix M^h is shown in Figure 1. This matrix corresponds to $N = 3, n = 17$ and M^h is a 17×17 block matrix in which each block is a 3×3. Figure 1 shows that a blocked lower triangular matrix is not necessarily a lower triangular matrix. It is worth to note that the first block (top-left) is a diagonal matrix.

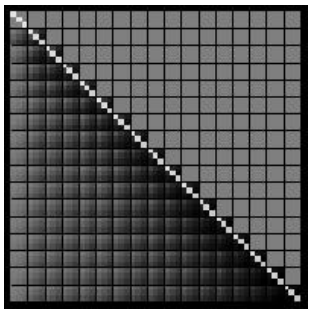


FIGURE 1. A simple view of the blocked lower triangular matrix results from the numerical scheme (2.12)

3. Main Results

Lemma 3.1. *For any matrix* $G = [g^{ij}(h)]$ *, if* $g^{ij}(h) = O(\delta(h))$ *as* $h \to 0$*, then for* $p = 1, 2, \infty$

$$\|G\|_p = O(\delta(h)),$$

where $\|.\|_p$ *denotes the standard matrix operator norm.*

Proof. According to the definitions $\|G\|_1 = \max_j \sum_i |g^{ij}|$ and $\|G\|_\infty = \max_i \sum_j |g^{ij}|$, it is clear that the result is true for $\|.\|_1$ and $\|.\|_\infty$. The result for $\|.\|_2$ is a consequence of the inequality

$$\|G\|_2 \leq (\|G\|_1.\|G\|_\infty)^{\frac{1}{2}}.$$

For more details see [4].

Lemma 3.2. *For the sufficiently smooth functions* $\boldsymbol{f}$ *and* $\boldsymbol{K}$ *the following results can be derived*

I.: $\boldsymbol{A}_{ij} = O(h^{1-\nu}),\ i = 0,\ 1, ..., n,\ , j = 0, 1, ..., i.$

II.: *For the sufficiently small* h, $\|\boldsymbol{K}_{ii}\boldsymbol{A}_{ii}\|_2 \leq 1, i = 0, 1, ..., n.$

III.: *The matrices* $I_N - \boldsymbol{K}_{ii}\boldsymbol{A}_{ii}, i = 0, 1, ..., n$ *are non-singular.*

IV.: *The numerical scheme (2.12) has a unique solution.*

Proof. The proof of results I, II and IV are straightforward and thus omitted. The statement III is a consequence of the geometric series. For more details we refer the readers to [4].

Lemma 3.3. *Suppose that the input data* $\boldsymbol{f}$ *and* $\boldsymbol{K}$ *are sufficiently smooth functions, then*

$$\|\boldsymbol{A}_{ij}\|_2 = O(h^{1-\nu}), \|\boldsymbol{B}_{ij}\|_2 = O(h^{1-\nu}).$$

Proof. The proof of this result is a consequence of the definition $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$ in (2.10).

Lemma 3.4. *For the following system of weakly singular VIE*

$$\boldsymbol{u}(t) = \boldsymbol{f}(t) + \mu \int_0^t \boldsymbol{K}(t, \tau)\boldsymbol{\Phi}(t, \tau)\boldsymbol{u}(\tau)d\tau. \tag{3.1}$$

where μ *is a real number, and* $\boldsymbol{f}$ *and* $\boldsymbol{K}$ *are sufficiently smooth functions, if* h *and* μ *are sufficiently small, then (2.13) is a diagonally dominant system.*

Proof. The proof of this lemma is based on definitions (2.10) and asymptotic behavior of $\mathbf{A}_{ij}$ and $\mathbf{B}_{ij}$.

The following lemma gives the order of truncation error corresponding to STEP 1.

Lemma 3.5. *If $\boldsymbol{u}$ and $\boldsymbol{K}$ are sufficiently smooth functions, then*

$$\| \int_{t_{j-1}}^{t_j} \boldsymbol{K}(t_i,\tau)\boldsymbol{\Phi}(t_i,\tau)\boldsymbol{u}(\tau)d\tau - \boldsymbol{K}_{i,j} \int_{t_{j-1}}^{t_j} \boldsymbol{\Phi}(t_i,\tau)\boldsymbol{u}(\tau)d\tau \|_2 = O(h^2). \tag{3.2}$$

where $\|.\|_2$ is the standard matrix operator 2-norm.

Proof. The proof is a direct consequence of the Taylor expansion theorem with integral reminder term (see [1], theorem 9.29) and using the inequality maintained in the proof of lemma 1.

Remark 3.6. In STEP 1, instead of using approximations

$$\mathbf{K}(t_i,\tau) \approx \mathbf{K}(t_i,t_j), \quad \text{or} \quad \mathbf{K}(t_i,\tau) \approx \mathbf{K}(t_i,t_{j-1}), \quad \tau \in [t_{j-1},t_j],$$

one can use more accurate interpolation schemes to obtain a method with a higher order truncation error. However, in this work we preferred a weaker approximation technique to avoid increasing the run-time.

The following lemma gives the order of truncation error corresponding to the approximation in STEP 2.

Lemma 3.7. *Suppose that $\boldsymbol{u} \in C^2([0,T],\mathbb{R}^N)$. Then*

1.: *For case II*

$$\| \int_{t_{j-1}}^{t_j} \boldsymbol{Q}_{ij}(\tau)\boldsymbol{u}'(\tau)d\tau \; - \frac{\boldsymbol{u}_j - \boldsymbol{u}_{j-1}}{h} \int_{t_{j-1}}^{t_j} \boldsymbol{Q}_{ij}(\tau)d\tau \|_2 = \boldsymbol{c}_1 h^{4-\nu} + O(h^{5-\nu}). \tag{3.3}$$

2.: *For case III,*

$$\| \int_{t_{j-1}}^{t_j} \boldsymbol{Q}_{ij}(\tau)\boldsymbol{u}'(\tau)d\tau \; - \frac{\boldsymbol{u}_j - \boldsymbol{u}_{j-1}}{h} \int_{t_{j-1}}^{t_j} \boldsymbol{Q}_{ij}(\tau)d\tau \|_2 = \boldsymbol{c}_1 h^2 |logh| + O(h^2), \tag{3.4}$$

where $\|.\|_2$ is the standard vector 2-norm and c_1 is a constant not depending on h.

Proof. The proof is a direct consequence of the Taylor expansion theorem and the proof of lemma 4.5 given in [12].

Theorem 3.8 (Main Theorem)**.** *Consider the linear operator* $\boldsymbol{\Psi}$ *as*

$$\boldsymbol{\Psi}\boldsymbol{u} = \int_0^t \boldsymbol{K}(t,\tau)\boldsymbol{\Phi}(t_i,\tau)\boldsymbol{u}(\tau)d\tau. \tag{3.5}$$

Let $\boldsymbol{\Psi}_n^h$ *be the corresponding discrete operator defined by*

$$\boldsymbol{\Psi}_n^h\boldsymbol{u} = \sum_{j=1}^{n} \boldsymbol{K}_{ij}(\boldsymbol{A}_{ij}\boldsymbol{u}_j + \boldsymbol{B}_{ij}\boldsymbol{u}_{j-1}), \tag{3.6}$$

then the asymptotic expansion of truncation errors for cases I and II have the form

1.: *For the case I and II:*

$$\|(\boldsymbol{\Psi} - \boldsymbol{\Psi}_n^h)\boldsymbol{u}\|_2 = \begin{cases} \boldsymbol{c}_1 h + \boldsymbol{c}_2 h^2 + O(h^3), & \text{for} \quad \nu \leq 0, \\ \boldsymbol{c}_1 h + \boldsymbol{c}_2 h^2 + \boldsymbol{c}_3 h^{3-\nu} + O(h^3), & \text{for} \quad 0 < \nu < 1. \end{cases} \tag{3.7}$$

2.: *For the case III:*

$$\|(\boldsymbol{\Psi} - \boldsymbol{\Psi}_n^h)\boldsymbol{u}\|_2 = \boldsymbol{c}_1 h + \boldsymbol{c}_2 h^2 |logh| + O(h^2), \tag{3.8}$$

where $\|.\|_2$ *is the standard vector 2-norm and* c_1, c_2 *and* c_3 *are constants independent of* h.

Proof. The proof is a consequence of the main theorems in [12] and lemma 3.1.

Remark 3.9. In lemma 3.5, lemma 3.7 and theorem 3.8, we can use general $\|.\|_p$; the standard matrix, vector p- norms for $p = 1, 2, \infty$.

4. Numerical Experiments

In this section the accuracy and ability of the present finite difference method for finding the smooth solution for a system of linear VIEs are illustrated by using two types of such equations, namely regular and weakly singular systems of VIEs. To solve weakly singular equations, our numerical experiments in Case I show that the accuracy of the finite difference method as a function of ν is a decreasing function, i.e. the error of approximate solution increases as ν increase from 0 to 1. Moreover, the comparison between the results in Cases I and II indicates that this method is more accurate in Case II than the Case I. Therefore, we restrict our numerical examples to Case I, though these assertions cannot

be true in general. Let $E(\mu) = \|\mathbf{U} - \mathbf{U}^h\|_\infty$ be the absolute error of the approximation, and

$$\|\mathbf{\Psi}\|_W = |\mu| \max_{0\le t\le 1} \| \int_0^t \mathbf{K}(t,\tau)d\tau\|_2, \tag{4.1}$$

denotes the Lipschitz factor corresponding to the integral operator [2, 13]

$$\mathbf{\Psi u} = \int_0^t \mathbf{K}(t,\tau)\mathbf{u}(\tau)d\tau. \tag{4.2}$$

Also, let $C(\mu)$ be the condition number of the matrix M^h defined in (2.13), $\tilde{\mathbf{U}}^h$ defined by

$$\tilde{\mathbf{U}}^h = 2\mathbf{U}^{\frac{h}{2}} - \mathbf{U}^h, \tag{4.3}$$

shows the extrapolated solution and $\tilde{E}(\mu) = \|\mathbf{U} - \tilde{\mathbf{U}}^h\|_\infty$ denotes the extrapolation error. In all tables, the numerical values for errors, condition numbers and Lipschits factors are rounded to three significant digits, and the last column shows that the matrixt M^h is either a diagonally dominant (DD) matrix or not.

Example 4.1. In this example we consider a regular VIE and compare its exact solution with the finite difference solution. This example is given in [16]. Consider the following system of integral equations

$$\begin{cases} u^{(1)}(t) = f^{(1)}(t) + \int_0^t (t-\tau)^3 u^{(1)}(\tau)d\tau + \int_0^t (t-\tau)^2 u^{(2)}(\tau)d\tau \\ u^{(2)}(t) = f^{(2)}(t) + \int_0^t (t-\tau)^4 u^{(1)}(\tau)d\tau + \int_0^t (t-\tau)^3 u^{(2)}(\tau)d\tau \end{cases} \tag{4.4}$$

The exact solution of this system of equations may be derived as

$$u^{(1)}(t) = 1 + t^2, u^{(2)}(t) = 1 + t - t^3,$$

where

$$f^{(1)}(t) = 1 + t^2 - t^3/3 - t^3/3, f^{(2)}(t) = 1 + t - 4t^3/3 - t^4/4 - t^5/12 + t^7/140.$$

Assume $\mathbf{u}(t) = [u^{(1)}(t), u^{(2)}(t)]^T$, $\mathbf{f}(t) = [f^{(1)}(t), f^{(2)}(t)]^T$ and

$$\mathbf{K}(t,\tau) = \begin{pmatrix} (t-\tau)^3 & (t-\tau)^2 \\ (t-\tau)^4 & (t-\tau)^3 \end{pmatrix},$$

then the system (4.4) take the form

$$\mathbf{u}(t) = \mathbf{f}(t) + \int_0^t \mathbf{K}(t,\tau)\mathbf{u}(\tau)d\tau. \tag{4.5}$$

For a more detailed discussion of the finite difference method, instead of equation (4.5) we consider following equation

$$\mathbf{u}(t) = \mathbf{f}(t) + \mu \int_0^t \mathbf{K}(t,\tau)\mathbf{u}(\tau)d\tau, \tag{4.6}$$

where μ is an arbitrary parameter. The exact solution for this problem considered as

$$\mathbf{u}(t) = [1+t^2, 1+t-t^3]^T, \mathbf{u} \in C^2([0,1], \mathbb{R}^2).$$

The numerical results for $\mu = 1, n = 32, 64, 128, 256$ illustrated in Table 1, and for $\mu = 2, n = 32, 64, 128, 256$ given in Table 2.

TABLE 1. The amounts of $C(\mu)$, $L(\mu)$, $E(\mu)$, $\tilde{E}(\mu)$ and DD when $\mu = 1$ and $n = 32, 64, 128, 256$ for Example 4.1.

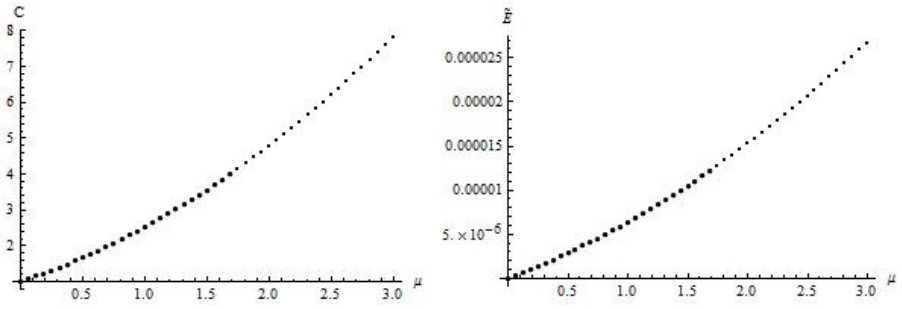
n	$C(\mu)$	$L(\mu)$	$E(\mu)$	$\tilde{E}(\mu)$	DD
32	2.47	0.583	0.0189	0.000395	True
64	2.50	0.583	0.00951	0.000100	True
128	2.51	0.583	0.00477	0.0000253	True
256	2.52	0.583	0.00239	6.35E-6	True

TABLE 2. The amounts of $C(\mu)$, $L(\mu)$, $E(\mu)$, $\tilde{E}(\mu)$ and DD when $\mu = 2$ and $n = 32, 64, 128, 256$ for Example 4.1.

n	$C(\mu)$	$L(\mu)$	$E(\mu)$	$\tilde{E}(\mu)$	DD
32	4.56	1.17	0.0391	0.000933	False
64	4.72	1.17	0.0196	0.000239	False
128	4.76	1.17	0.00985	0.0000604	False
256	4.77	1.17	0.00493	0.0000152	False

Figure 2 demonstrates the graph of $(L(\mu), C(\mu)), 0 \leq \mu \leq 3$ in the coordinate system (L, C) and the graph of $(L(\mu), \tilde{E}(\mu))$ in the coordinate system $(L, \tilde{E})$.

In Figure 2a the points with larger point size indicate that for corresponding value of the variable μ, the operator $\mathbf{K}$ is a contraction mapping. Similarly, the points with the larger point size in Figure 2b illustrate that the matrix M^h is a diagonally dominant matrix. This property verifies the result of Lemma 3.3.



a. The graph of $C(\mu)$ b. The graph of $\tilde{E}(\mu)$

FIGURE 2. The graph of the absolute error $\tilde{E}(\mu)$ (a) and the condition numbers $C(\mu)$ (b) for Example 4.1..

Example 4.2. In this example, we examine the proposed finite difference method to solve a weakly singular system of the Volterra integral equation that satisfies a smooth solution. Consider the system

$$\mathbf{u}(t) = \mathbf{f}(t) + \mu \int_0^t \mathbf{K}(t,\tau)\mathbf{\Phi}(t,\tau)\mathbf{u}(\tau)d\tau, \tag{4.7}$$

where

$$\mathbf{K}(t,\tau) = \begin{pmatrix} 1 & t-\tau \\ t-\tau & 1 \end{pmatrix}, \mathbf{\Phi}(t,\tau) = (t-\tau)^{-1/2} I_2,$$

$$\mathbf{f}(t) = [t - 4t^{3/2}(35+4t^2)/315, t^2 - 4t^{5/2}/9]^t$$

and I_2 shows the 2×2 identity matrix. The exact solution of this equation can be derived as

$$\mathbf{u}(t) = [t, t^2]^T, \mathbf{u} \in C^2([0,1], \mathbb{R}^2).$$

In this example μ is an arbitrary parameter. Using the notation given in example(4.1), the numerical results for $\mu = 1/4, n = 32, 64, 128, 256, 512$ illustrated in Table 3, and for $\mu = 1/2, n = 32, 64, 128, 256, 512$ given in Table 4.

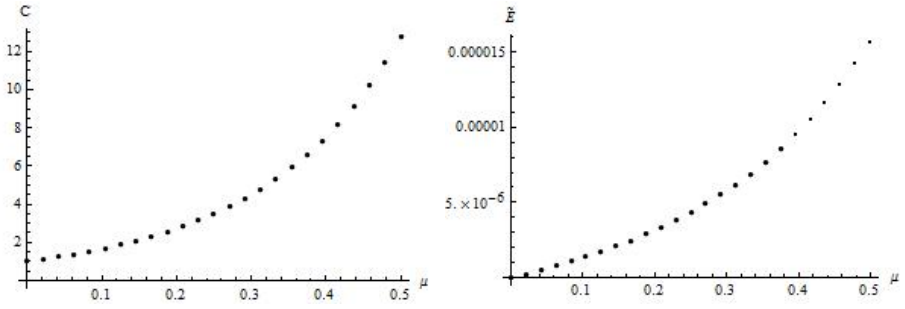
Similar to Example 4.1, the graph of $L(\mu), 0 \le \mu \le 1$ in the coordinate system (μ, L) and the graph of $\tilde{E}(\mu), 0 \le \mu \le 1$ in the coordinate system $(\mu, \tilde{E})$ are illustrated in Figure 3. In Figure 3a the points with larger point size indicate the contraction mapping property of the operator $\mathbf{K}$ for the corresponding values of μ. Furthermore, the points with the larger point size in Figure 3b shows that the matrix M^h is diagonally dominant.

TABLE 3. The amounts of $C(\mu)$, $L(\mu)$, $E(\mu)$, $\tilde{E}(\mu)$ and DD when $\mu = \frac{1}{4}$ and $n = 32, 64, 128, 256$ for Example 4.2.

n	$C(\mu)$	$L(\mu)$	$E(\mu)$	$\tilde{E}(\mu)$	DD
32	3.34	0.375	0.00340	0.000327	True
64	3.41	0.375	0.00175	0.000109	True
128	3.45	0.375	0.000895	0.0000369	True
256	3.48	0.375	0.000454	0.0000126	True
512	3.50	0.375	0.000229	4.33E-6	True

TABLE 4. The amounts of $C(\mu)$, $L(\mu)$, $E(\mu)$, $\tilde{E}(\mu)$ and DD when $\mu = \frac{1}{2}$ and $n = 32, 64, 128, 256$ for Example 4.2.

n	$C(\mu)$	$L(\mu)$	$E(\mu)$	$\tilde{E}(\mu)$	DD
32	11.9	0.750	0.0109	0.00121	False
64	12.3	0.750	0.00563	0.000403	False
128	12.5	0.750	0.00289	0.000135	False
256	12.7	0.750	0.00147	0.0000458	False
512	12.8	0.750	0.000741	0.0000156	False



a. The graph of $C(\mu)$ b. The graph of $\tilde{E}(\mu)$

FIGURE 3. The graph of the absolute error $\tilde{E}(\mu)$ (a) and the condition numbers $C(\mu)$ (b) for Example 4.2.

All numerical computations in this work are generated using Mathematica (version 5.5) and Matlab codes, in a Linux Cluster environment, at the Math Computing Center, Institute for the Research in Fundamental Science IPM.

5. Conclusion

In this paper we presented a fast numerical scheme for generating the smooth solutions of VIEs having regular or weakly singular (non-logarithmic) kernels. It is apparent that the order of the truncation is lower than the ones in the previously reported methods. However, it is shown that the numerical results can be improved by applying the extrapolation technique. For proper comparisons of this method with other methods, we need to develop a convergence theory. This could be investigated in future works.

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