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α-TYPE SHORT MODULES

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Abstract. In this paper, we first consider the concept of type Noetherian dimension of a module such as M, which is the dual of type krull dimension, denoted by $tn\text{-}\dim M$, and defined to be the codeviation of the poset of the type submodules of M, then we dualize some of the basic results of type Krull dimension for type Noetherian dimension. In the following, we introduce the concept of α-type short modules (i.e., for each type submodule N of M, either $n\text{-}\dim \frac{M}{N} \leq \alpha$ or $tn\text{-}\dim N \leq \alpha$ and α is the least ordinal number with this property), and extend some of the basic results of α-short modules to α-type short modules. In particular, it is proved that if M is an α-type short module, then it has type Noetherian dimension and $tn\text{-}\dim M = \alpha$ or $tn\text{-}\dim M = \alpha + 1$.

1. Introduction

The Krull dimension of a module M, denoted by $k\text{-}\dim M$ and measures its deviation from being Artinian, was first introduced by Gabriel and Rentschler (for finite ordinals), see [17]. Later this definition was extended to infinite ordinals by Krause in 1970, see [14]. Lemonnier introduced the concept of the deviation of an arbitrary poset $(E, \leq)$, similar to the concept of Krull dimension of modules, see [16]. We remind the reader that the dual of Krull dimension, measures the deviation of a module from being Noetherian. We

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should emphasize, for the sake of record and the reader, that the dual of Krull dimension of a module was first named Noetherian dimension by Karamzadeh in his Ph.D. thesis at Exeter university, England, see [11], and later it is studied in [4, 5, 6, 8, 9, 12, 13, 15, 20, 21]. Let us denote the dual of Krull dimension of a module M by n-dim M. The module M satisfies the ascending chain condition(ACC, for short) on type submodules if and only if t.dim $M < \infty$, see [3, Proposition 4.1.12 (2)]. Motivated by this fact, one is tempted to extend it to an ascending chain of type submodules of M. To this end, we first introduce and study the concept of type Noetherian dimension of a module M, which is the dual of type krull dimension, see [19]. This dimension is defined to be the codeviation of the poset of the type submodules of M. In some sense, it measures how far the type Noetherian dimension is from finite type dimension. We briefly study this dimension and observe that if M has Noetherian dimension, then it has type Noetherian dimension and tn-dim $M \leq n$-dim M. Bilhan and Smith [2], introduced short modules i.e., for every submodule N of M, either N is Noetherian or $\frac{M}{N}$ is Noetherian. Karamzadeh et al. [4], by extending this concept to every ordinal number α, defined α-short modules. They called a module M, α-short whenever for every submodule N of M, either n-dim $N \leq \alpha$ or n-dim $\frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. In addition to this extension, they obtained all the results of [2] in the special case. In particular, they showed that every α-short module has Noetherian dimension equal to α or $\alpha+1$. Shirali [18], extended α-short modules to α-small short modules and later she et al. [10], extended α-short modules to α-parallel short modules. In this article, we introduce and study the concept of α-type short modules. Using this concept, we extend some of the basic results of α-short modules to α-type short modules. Let us give a brief sketch of this article. After reviewing some necessary preliminaries, in Section 3, we introduce the concept of type Noetherian dimension of an R-module M and obtain some of the basic results about modules with this dimension. In the following, we introduce the concept of type atomic modules, and we prove that if M is an R-module with type Noetherian dimension, then M has a non-zero type quotient module which is type atomic. In Section 4, we first introduce and study the concept of α-type short modules, and we show that if M is an α-type short module, then either tn-dim $M = \alpha$ or tn-dim $M = \alpha+1$. Finally, we study the concept of α-almost type Noetherian modules, and we prove that if M is an α-almost type Noetherian module, then M has type Noetherian dimension and tn-dim $M \leq \alpha+1$.

We recall that an R-module M is called α-atomic, where α is an ordinal, if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$, for every proper submodule N of M. An R-module M is said to be atomic if it is α-atomic, for some α. Throughout this article, all rings are associative with identity and all modules are unitary right modules. It is convenient that, when we are dealing with the latter dimensions, we may begin our list of ordinals with -1. $N \subseteq M$ (resp., $N \subset M$) will mean N is a submodule (resp., a proper submodule) of M. For all concepts and basic properties of rings and modules which are not defined in this paper, we refer the reader to [1, 7].

2. Preliminaries

This section contains some preliminary results that are needed in the sequel, most of which are in [3].

First, we recall the following definitions.

Definition 2.1. Two modules A and B are orthogonal, written as $A \perp B$, if they do not have non-zero isomorphic submodules. Modules A_1 and A_2 are parallel, denoted as $A_1 \parallel A_2$, if there does not exist a $0 \neq C_2 \subseteq A_2$ with $A_1 \perp C_2$, and also there does not exist a $0 \neq C_1 \subseteq A_1$ such that $C_1 \perp A_2$. An equivalent definition of $A_1 \parallel A_2$ is that for any $0 \neq C_1 \subseteq A_1$, there exist $0 \neq aR \subseteq C_1$ and $0 \neq bR \subseteq A_2$ with $aR \cong bR$, and dually for any $0 \neq C_2 \subseteq A_2$, there exist $0 \neq aR \subseteq A_1$ and $0 \neq bR \subseteq C_2$ with $aR \cong bR$. Also, a module M is called atomic if all of its non-zero submodules are parallel to each other.

Definition 2.2. A submodule A of an R-module M is called a type submodule, denoted as $A \subseteq_t M$, if the following equivalent conditions hold:

(1) If $A \subseteq B \subseteq M$ with $A \parallel B$, then $A = B$.

(2) If $A \subset B \subseteq M$, then $A \perp X$ for some $0 \neq X \subseteq B$.

Lemma 2.3. *Let M be an R-module with submodules A and B.*

(1) *$M \neq 0$ is an atomic module if and only if (0) and M are the only type submodules of M.*

(2) *If $A \subseteq_t M$ and $A \subseteq B$, then $A \subseteq_t B$.*

(3) *If $A \subseteq_t B \subseteq_t M$, then $A \subseteq_t M$.*

Lemma 2.4. *Let M be an R-module and A a submodule of M. Let $A \subseteq_t M$ and $A \subseteq B \subseteq M$. Then $\frac{B}{A} \subseteq_t \frac{M}{A}$ if and only if $B \subseteq_t M$.*

Definition 2.5. An R-module M has finite type dimension n, denoted by $t.\dim M = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. If no such n exists, we say that the type dimension of M is infinite, and write $t.\dim M = \infty$. If $M = 0$, then $t.\dim M = 0$.

For a module M, $t.\dim M = \infty$ if and only if there exist an infinite number of pairwise orthogonal non-zero submodules of M.
Let us continue with the following well-known and important result, see [3, Proposition 4.1.12(2)].

Proposition 2.6. *The following statements are equivalent for an R-module M.*

(1) $t.\dim M < \infty$.
(2) *M has the ascending chain condition (briefly, ACC) on type submodules.*
(3) *M has the descending chain condition (briefly, DCC) on type submodules.*

3. The dual of type Krull dimension and its properties

In this section, we consider the concept of the dual of type Krull dimension of an R-module M, which is a Noetherian-like dimension extension of the concept of *ACC* over type submodules. In other words, it is the codeviation of the poset of type submodules of M. Also, we introduce the concept of type atomic modules, which is similar to the concept of atomic modules in [12].
Next, we give our definition of type Noetherian dimension.

Definition 3.1. Let M be an R-module. The type Noetherian dimension of M, denoted by $tn\text{-}\dim M$, is defined by transfinite recursion as follows: If $M = 0$, $tn\text{-}\dim M = -1$. If α is an ordinal number and $tn\text{-}\dim M \not< \alpha$, then $tn\text{-}\dim M = \alpha$ provided there is no infinite ascending chain of type submodules of M such as $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ such that for each $i = 1, 2, \cdots$, $tn\text{-}\dim \frac{M_{i+1}}{M_i} \not< \alpha$. In other words, $tn\text{-}\dim M = \alpha$, if $tn\text{-}\dim M \not< \alpha$ and for each chain of type submodules of M such as $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ there exists an integer t, such that for each $i \geq t$, $tn\text{-}\dim \frac{M_{i+1}}{M_i} < \alpha$. A ring R has type Noetherian dimension, if as an R-module has type Noetherian dimension. It is possible that there is no ordinal α such that $tn\text{-}\dim M = \alpha$, in this case we say M has no type Noetherian dimension.

If $tn\text{-}\dim M > \alpha$, there exists an infinite ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of type submodules of M such that $tn\text{-}\dim \frac{M_{i+1}}{M_i} \geq \alpha$, for all i.

Clearly, $tn\text{-}\dim M = 0$ if and only if M satisfies ACC on its type submodules. So $tn\text{-}\dim M = 0$ if and only if $t.\dim M < \infty$, by Proposition 2.6. In this case, M satisfies DCC on its type submodules.

Remark 3.2. [19, Remark 3.3] Every module with Krull dimension has type Krull dimension.

Theorem 3.3. [19, Theorem 3.25] *An R-module M has type Krull dimension if and only if it has type Noetherian dimension.*

Using the above facts and this fact that every module has Krull dimension if and only if it has Noetherian dimension, the following fact is evident.

Lemma 3.4. *If M is an R-module with Noetherian dimension, then it has type Noetherian dimension and $tn\text{-}\dim M \leq n\text{-}\dim M$.*

The proofs of the following results are just a minor variant of the familiar argument for type Krull dimension, see [19].

Lemma 3.5. *Let M be an R-module with type Noetherian dimension. Then for each type submodule N of M, N has type Noetherian dimension and $tn\text{-}\dim N \leq tn\text{-}\dim M$.*

Lemma 3.6. *Let M be an R-module with type Noetherian dimension. Then for each type submodule N of M, $\frac{M}{N}$ has type Noetherian dimension and $tn\text{-}\dim \frac{M}{N} \leq tn\text{-}\dim M$.*

Proposition 3.7. *Let M be an R-module. If every type submodule of M has type Noetherian dimension, then so does M and $tn\text{-}\dim M = \sup\{tn\text{-}\dim N : N \subseteq_t M\}$.*

Proposition 3.8. *Let M be an R-module. If for each type submodule N of M, $\frac{M}{N}$ has type Noetherian dimension, then so does M and $tn\text{-}\dim M \leq \sup\{tn\text{-}\dim \frac{M}{N} : N \subseteq_t M\} + 1$.*

Theorem 3.9. *Let M be an R-module. If for each type submodule N of M, N or $\frac{M}{N}$ has type Noetherian dimension, then so does M.*

Proposition 3.10. *Let M be an R-module. If for every type submodule N of M, N has Noetherian dimension, then M has type Noetherian dimension and $tn\text{-}\dim M \leq \sup\{n\text{-}\dim N : N \subseteq_t M\} + 1$.*

Corollary 3.11. *Let M be an R-module. If for every type submodule N of M, N has Noetherian dimension and $n\text{-}\dim N < \alpha$, then $tn\text{-}\dim M \leq \alpha + 1$.*

Proposition 3.12. *Let M be an R-module. If for every type submodule N of M, $\frac{M}{N}$ has Noetherian dimension, then M has type Noetherian dimension and $tn\text{-}\dim M \leq \sup\{n\text{-}\dim \frac{M}{N} : N \subseteq_t M\} + 1$.*

We recall that a submodule E of an R-module M is said to be essential if $E \cap B \neq 0$, for each non-zero submodule B of M. Also, a submodule K of M is called closed, if N is a submodule of M such that K is an essential submodule of N, then $N = K$.

Theorem 3.13. *Let M be an R-module.*

(1) *Every decomposable submodule N of M has $n\text{-}\dim N \leq \alpha$.*
(2) *Every closed submodule N of M has $tn\text{-}\dim N \leq \alpha$.*
(3) *Every type submodule N of M has $tn\text{-}\dim N \leq \alpha$.*
(4) *M has type Noetherian dimension.*

Then we have $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

Next, we give our definition of type atomic modules, which is similar to the concept of atomic modules.

Definition 3.14. An R-module M is called α-type atomic if $tn\text{-}\dim M = \alpha$ and for every proper type submodule N of M, $tn\text{-}\dim N < \alpha$. M is called type atomic if it is an α-type atomic for some α.

We note that an R-module M is 0-type atomic if and only if it has no non-zero type submodule.

Definition 3.15. Let N be a type submodule of an R-module M. Then $\frac{M}{N}$ is called a type quotient module of M.

Proposition 3.16. *Let M be an R-module with type Noetherian dimension. Then M has a non-zero type quotient module which is type atomic.*

Proof. Assume that every non-zero type quotient module of M is non-type atomic and seek a contradiction. Let $\frac{M}{N}$ be a non-zero type quotient module of M with the least type Noetherian dimension. Clearly, if $\frac{M}{N}$ has a type quotient module, then so does M, by

Lemma 2.4. Thus, without loss of generality we may assume that M has the least type Noetherian dimension among its non-zero type quotient modules. Now, let $tn\text{-}\dim M = \alpha$, then by our assumption, there exists a type submodule, N_1 say of M with $tn\text{-}\dim N_1 = \alpha$. But $\frac{M}{N_1}$ is not type atomic and $tn\text{-}\dim \frac{M}{N_1} = \alpha$. Similarly, there is a type submodule $\frac{N_2}{N_1}$ of $\frac{M}{N_1}$ with $tn\text{-}\dim \frac{N_2}{N_1} = \alpha$. If we repeat this process, we obtain an infinite ascending chain of type submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ in M with $tn\text{-}\dim \frac{N_{i+1}}{N_i} = \alpha$ for each i, which is a contradiction. □

4. α-type short modules

In this section, we introduce and study the concept of α-type short modules. Using this concept, we extend some of the basic results of α-short modules, see [4], to α-type short modules. Also, we introduce the concept of α-almost type Noetherian module and investigate the properties of it over an arbitrary ring R.
We begin with the following definition.

Definition 4.1. An R-module M is called α-type short if for each type submodule N of M, either $n\text{-}\dim \frac{M}{N} \leq \alpha$ or $tn\text{-}\dim N \leq \alpha$ and α is the least ordinal number with this property.

Clearly, if M is a -1-type short module, then M has not non-zero proper type submodule. So M is an atomic module, by Lemma 2.3(1).

Lemma 4.2. *If M is an R-module with $tn\text{-}\dim M = \alpha$, then M is a β-type short module for some $\beta \leq \alpha$.*

Proof. By Lemma 3.5, for every type submodule N of M, $tn\text{-}\dim N \leq tn\text{-}\dim M = \alpha$. Hence, M is a β-type short module for some $\beta \leq \alpha$. □

Lemma 4.3. *Let M be an α-type short module. Then every type submodule N of M is a β-type short module for some $\beta \leq \alpha$.*

Proof. Let L be a type submodule of N, then $L \subseteq_t M$, by Lemma 2.3(3). So, either $n\text{-}\dim \frac{M}{L} \leq \alpha$ or $tn\text{-}\dim L \leq \alpha$, hence either $n\text{-}\dim \frac{N}{L} \leq n\text{-}\dim \frac{M}{L} \leq \alpha$ or $tn\text{-}\dim L \leq \alpha$. Therefore, N is a β-type short module for some $\beta \leq \alpha$. □

Lemma 4.4. *Let M be an α-type short module. Then every type quotient module of M is a β-type short module for some $\beta \leq \alpha$.*

Proof. Let $\frac{M}{N}$ be a type quotient module of M and $\frac{L}{N} \subseteq_t \frac{M}{N}$. By Lemma 2.4, $L \subseteq_t M$, so either $n\text{-dim}\, \frac{M/N}{L/N} = n\text{-dim}\, \frac{M}{L} \le \alpha$ or $tn\text{-dim}\, L \le \alpha$, hence either $n\text{-dim}\, \frac{M}{L} \le \alpha$ or $tn\text{-dim}\, \frac{L}{N} \le tn\text{-dim}\, L \le \alpha$. Therefore, $\frac{M}{N}$ is a β-type short module for some $\beta \le \alpha$. □

Lemma 4.5. *Let M be an α-type short module. Then M has type Noetherian dimension and $tn\text{-dim}\, M \ge \alpha$.*

Proof. According to the assumption, either $n\text{-dim}\, \frac{M}{N} \le \alpha$ or $tn\text{-dim}\, N \le \alpha$, for every type submodule N of M. By Lemma 3.4 and Theorem 3.9, M has type Noetherian dimension, so $tn\text{-dim}\, M \ge \alpha$, by Lemma 4.2. □

The following result is now immediate.

Corollary 4.6. *An R-module M has type Noetherian dimension if and only if it is an α-type short module for some ordinal number α.*

In view of the previous lemma and Theorem 3.3, it can be said that every α-type short module has type Krull dimension.

Proposition 4.7. *Let M be an α-type short module. Then either $tn\text{-dim}\, M = \alpha$ or $tn\text{-dim}\, M = \alpha + 1$.*

Proof. In view of Lemma 4.5, we have $tn\text{-dim}\, M \ge \alpha$. If $tn\text{-dim}\, M \ne \alpha$, then $tn\text{-dim}\, M \ge \alpha+1$. Let $M_1 \subseteq M_2 \subseteq \cdots$ be any ascending chain of type submodules of M. Since M is an α-type short module, either $n\text{-dim}\, \frac{M}{M_i} \le \alpha$ or $tn\text{-dim}\, M_i \le \alpha$, for all i. If there exists some n such that $n\text{-dim}\, \frac{M}{M_n} \le \alpha$, then $n\text{-dim}\, \frac{M_{i+1}}{M_i} \le n\text{-dim}\, \frac{M}{M_i} = n\text{-dim}\, \frac{M/M_n}{M_i/M_n} \le n\text{-dim}\, \frac{M}{M_n} \le \alpha$, for each $i \ge n$. By Lemma 3.4, $tn\text{-dim}\, \frac{M_{i+1}}{M_i} \le n\text{-dim}\, \frac{M_{i+1}}{M_i} \le \alpha$, so $tn\text{-dim}\, M \le \alpha + 1$. Otherwise, $tn\text{-dim}\, M_n \le \alpha$, for each n. But, by Lemma 2.3(2,3), all elements of this chain are type to each other and they have type Noetherian dimension, so by Lemma 3.6, $\frac{M_{i+1}}{M_i}$ has type Noetherian dimension, for all i, and $tn\text{-dim}\, \frac{M_{i+1}}{M_i} \le tn\text{-dim}\, M_{i+1} \le \alpha$, so $tn\text{-dim}\, M \le \alpha + 1$. Therefore, in any case $tn\text{-dim}\, M \le \alpha + 1$, and this shows that $tn\text{-dim}\, M = \alpha + 1$. □

Corollary 4.8. *If M is a 0-type short module, then either $tn\text{-dim}\, M = 1$ or M has finite type dimension. Also, if M is a -1-type short module, then either $M = 0$ or M has finite type dimension.*

Lemma 4.9. *Let M be a 0-type short module. Then either $tn\text{-}\dim M = 0$ or $\frac{M}{N}$ is Noetherian, for every non-trivial type submodule N of M.*

Proof. According to the assumption, either $n\text{-}\dim \frac{M}{N} \leq 0$ or $tn\text{-}\dim N \leq 0$. If $n\text{-}\dim \frac{M}{N} \leq 0$, then either $n\text{-}\dim \frac{M}{N} = -1$, which is impossible by hypothesis, or $n\text{-}\dim \frac{M}{N} = 0$, that is $\frac{M}{N}$ Noetherian. Otherwise, if $tn\text{-}\dim N \leq 0$, then either $tn\text{-}\dim N = -1$, which is impossible by hypothesis, or $tn\text{-}\dim N = 0$ that in this case, by Proposition 3.7, $tn\text{-}\dim M = 0$. □

Proposition 4.10. *Let M be an R-module with $tn\text{-}\dim M = \alpha$, where α is a limit ordinal. Then M is an α-type short module.*

Proof. By Lemma 4.2, M is a β-type short module for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Proposition 4.7, $tn\text{-}\dim M \leq \beta + 1 < \alpha$, which is a contradiction. Therefore, M is an α-type short module. □

Proposition 4.11. *Let M be an R-module and $tn\text{-}\dim M = \alpha$, where $\alpha = \beta + 1$. Then M is either α-type short or β-type short.*

Proof. By Lemma 4.2, M is a γ-type short module for some $\gamma \leq \alpha$. If $\gamma < \beta$, then by Proposition 4.7, $tn\text{-}\dim M \leq \gamma + 1 < \beta + 1$, which is a contradiction. Hence, $\beta \leq \gamma \leq \beta + 1 = \alpha$, which completes the proof. □

For the type atomic modules, we have the following proposition.

Proposition 4.12. *Let M be an α-type atomic R-module, where $\alpha = \beta + 1$. Then M is a β-type short module.*

Proof. Let N be a proper type submodule of M. Since M is an α-type atomic module, $tn\text{-}\dim N < \alpha$, so $tn\text{-}\dim N \leq \beta$. It follows that M is a γ-type short module for some $\gamma \leq \beta$. If $\gamma < \beta$, then by Proposition 4.7, $tn\text{-}\dim M \leq \gamma + 1 \leq \beta < \alpha$, which is a contradiction. Therefore, $\gamma = \beta$ and we are through. □

The following result, which is a trivial consequence of the previous proposition, shows that the converse of Proposition 4.10 is not true, in general.

Corollary 4.13. *Let M be an $\alpha + 1$-type atomic R-module, where α is a limit ordinal. Then M is an α-type short module but $tn\text{-}\dim M \neq \alpha$.*

Proposition 4.14. *Let M be an R-module such that $tn\text{-}\dim M = \alpha + 1$. Then M is either α-type short module or there exists a type submodule N of M such that $tn\text{-}\dim N = n\text{-}\dim \frac{M}{N} = \alpha + 1$.*

Proof. By Proposition 4.11, M is either α-type short or $\alpha + 1$-type short. Assume that M is not an α-type short module, hence there exists a type submodule N of M such that $tn\text{-}\dim N \geq \alpha + 1$ and $n\text{-}\dim \frac{M}{N} \geq \alpha + 1$. This shows that $tn\text{-}\dim N = \alpha + 1$ and $n\text{-}\dim \frac{M}{N} = \alpha + 1$ and we are through. □

Proposition 4.15. *Let M be an R-module. If every type submodule of M is an α-type short module, then $tn\text{-}\dim M \leq \alpha + 1$.*

Proof. Let N be a type submodule of M such that it is an α-type short module. By Proposition 4.7, $tn\text{-}\dim N \leq \alpha + 1$. So by Proposition 3.7, $tn\text{-}\dim M \leq \alpha + 1$. □

Proposition 4.16. *Let M be an R-module. If every type quotient module of M is an α-type short module, then $tn\text{-}\dim M \leq \alpha + 2$.*

Proof. Let N be a type submodule of M such that $\frac{M}{N}$ is an α-type short module. By Proposition 4.7, $tn\text{-}\dim \frac{M}{N} \leq \alpha + 1$. So by Proposition 3.8, $tn\text{-}\dim M \leq \alpha + 2$. □

The following result is a connection between α-short modules and α-type short modules.

Proposition 4.17. *Let M be an α-short module. Then M is a γ-type short module for some $\gamma \leq \alpha$.*

Proof. By [4, Corollary 1.5], every α-short module such as M has Noetherian dimension and for every type submodule N of M, either $n\text{-}\dim N \leq \alpha$ or $n\text{-}\dim \frac{M}{N} \leq \alpha$. So, by Lemma 3.4, either $tn\text{-}\dim N \leq n\text{-}\dim N \leq \alpha$ or $n\text{-}\dim \frac{M}{N} \leq \alpha$. It follows that M is a γ-type short module for some $\gamma \leq \alpha$. □

Next, we give our definition of α-almost type Noetherian modules, which is similar to the concept of α-almost Noetherian modules in [4].

Definition 4.18. An R-module M is called α-almost type Noetherian, if for each type submodule N of M, $n\text{-}\dim \frac{M}{N} < \alpha$ and α is the least ordinal number with this property.

Using Proposition 3.12, the following fact is easily proved.

Proposition 4.19. *Let M be an R-module. If M is an α-almost type Noetherian, then M has type Noetherian dimension and tn-*$\dim M \leq \alpha + 1$*.*

By virtue of Lemmas 2.3(3) and 2.4, one can easily prove the following result.

Lemma 4.20. *Let M be an R-module. If M is an α-almost type Noetherian, then for every type submodule N of M, N is a β-almost type Noetherian and $\frac{M}{N}$ is a γ-almost type Noetherian for some $\beta, \gamma \leq \alpha$.*

Proposition 4.21. *Let M be an R-module. If every type submodule of M is an α-almost type Noetherian, then tn-*$\dim M \leq \alpha + 1$*.*

Proof. Let N be a type submodule of M such that it is an α-almost type Noetherian, then for every type submodule L of N, n-$\dim \frac{N}{L} < \alpha$. Hence, by Proposition 3.12, tn-$\dim N \leq \alpha+1$, so by Proposition 3.7, M has type Noetherian dimension and tn-$\dim M \leq \alpha+1$. □

Proposition 4.22. *Let M be an R-module. If every type quotient module of M is an α-almost type Noetherian, then tn-*$\dim M \leq \alpha + 2$*.*

Proof. Let N be a type submodule of M such that $\frac{M}{N}$ is an α-almost type Noetherian. So, for every type submodule $\frac{L}{N}$ of $\frac{M}{N}$, n-$\dim \frac{M}{L} = n$-$\dim \frac{M/N}{L/N} < \alpha$. Thus by Proposition 3.12, tn-$\dim \frac{M}{N} \leq \alpha + 1$, hence by Proposition 3.8, M has type Noetherian dimension and tn-$\dim M \leq \alpha + 2$. □

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