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ON λ-PURE EXACT STRUCTURE

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Dedicated to Ali Rezaei Aliabad on his 70th birthday

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Abstract. Let λ be an infinite regular cardinal and $\mathcal{A}$ a locally λ-presentable additive category. We show that any λ-pure morphism (resp. λ-pure quotient) in $\mathcal{A}$ creates a kernel-cokernel pair. This implies that the class of all λ-pure kernel-cokernel pairs in $\mathcal{A}$ forms an exact structure. Additionally, we will describe λ-pure kernel-cokernel pairs in $\mathcal{A}$ and will prove that any λ-directed diagram of objects in $\mathcal{A}$ induces a canonical λ-pure kernel-cokernel pair.

1. Introduction

Exact categories provide a natural framework for developing relative homological algebra in non-abelian categories, which have a rich history in the literature (see [3], [5], [7], [11], [20], [4], [10]). Specifically, the concept of an exact additive category was introduced in [11] (and refined in [10]) by abstracting the essential properties of short exact sequences in abelian categories without requiring the existence of kernels and cokernels. Consequently, performing homological algebra in a given additive category $\mathcal{A}$ is closely related to the existence of a non-trivial exact structure on $\mathcal{A}$ (the trivial exact structure on $\mathcal{A}$ is defined by the class of all split kernel-cokernel pairs). In the case where $\mathcal{A}$ has both kernels and cokernels, the class $\mathcal{E}$ of all kernel-cokernel pairs in $\mathcal{A}$ defines an exact structure if and only if $\mathcal{A}$ is quasi-abelian (see [13] and [15]). Thus, in the general case, $\mathcal{E}$ fails to define an exact structure on $\mathcal{A}$ (see [12, Example 1]). Accordingly, the conditions under which a

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subclass of $\mathcal{E}$ induces a non-trivial exact structure on $\mathcal{A}$ are discussed in the literature (see [18], [14] and [6]). In this work, if $\mathcal{A}$ is a locally presentable additive category, we choose a special subclass $\mathcal{E}'$ of $\mathcal{E}$ and show that $(\mathcal{A}, \mathcal{E}')$ is an exact category. Moreover, we will give a characterization of elements in $\mathcal{E}'$.

Throughout this work, we assume that λ is **an infinite regular cardinal** and $\mathcal{A}$ is a **locally λ-presentable additive category**. By a λ-*directed diagram* in $\mathcal{A}$, we mean a commutative diagram indexed by a λ-directed partially ordered set, i.e. any subset with less than λ elements has an upper bound. λ-*directed colimits* are colimits of such diagrams. An object F in $\mathcal{A}$ is called λ-*presentable* if the functor $\mathrm{Hom}_{\mathcal{A}}(F,\text{-})$ commutes with λ-directed colimits. The category $\mathcal{A}$ is called locally λ-presentable if it is cocomplete and has a set S of λ-presentable objects such that any object of $\mathcal{A}$ is a λ-directed colimit of objects in S (see [1, Definitions 1.13, 1.17]). Recall from [1, Definition 2.27.] that a morphism $f : X \longrightarrow Y$ in $\mathcal{A}$ is called λ-*pure* if for any commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{f'} & C' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with λ-presentable objects C and C', there exists a morphism $g : C' \longrightarrow X$ such that $gf' = s$. It was shown in [1, Proposition 2.29] that λ-pure morphisms in $\mathcal{A}$ are monomorphisms ($f : X \longrightarrow Y$ is a monomorphism if for each pair of morphisms $g, h : Z \longrightarrow X$, $fg = fh$ implies $g = h$). Moreover, by [1, Proposition 2.3 (ii)], every λ-pure morphism in $\mathcal{A}$ is a λ-directed colimit of split monomorphisms. In the case in which $\lambda = \aleph_0$, $\aleph_0$-presentable objects are finitely presentable, locally presentable categories are locally finitely presentable and $\aleph_0$-pure morphisms are nothing else than pure monomorphisms.

It is known that for any cardinal $\gamma \geq \lambda$, γ-pure morphisms are λ-pure, but the converse is not necessarily true. In summary, while γ-pure morphisms are a generalization of pure morphisms, they do not always behave as expected for all infinite regular cardinals. This emphasizes the complexity of the theory of pure exact sequences in module categories. Consider the following example to illustrate this point.

Example 1.1. Let R be an associative ring with identity. It is well-known that the category R-Mod, consisting of all left R-modules, is a locally finitely presentable additive category. This means that

R-Mod is cocomplete and every object in R-Mod can be expressed as a directed colimit of finitely presentable objects. Now, assume for the sake of contradiction that for every cardinal $\gamma > \aleph_0$, any pure short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{1.1}$$

is γ-pure. Recall that a short exact sequence $\mathcal{E}$ is γ-pure if for any γ-presentable left R-module F, the sequence $\mathrm{Home}_R(F, \mathcal{E})$ remains exact. According to [1, Proposition 1.16], M'' is γ_0-presentable for some $\gamma_0 > \aleph_0$. This implies that the sequence (1.1) splits. Therefore, any pure short exact sequence of left R-modules splits, which is clearly a contradiction. Thus, there exists a large class of infinite regular cardinals $\gamma > \aleph_0$ such that (1.1) is not γ-pure.

Recall from [2, Definition 1] that a morphism $f : X \to Y$ in $\mathcal{A}$ is called a λ-*pure quotient* if for any λ-presentable object C, the morphism

$$\mathrm{Hom}_{\mathcal{A}}(C, X) \xrightarrow{\bar{f}} \mathrm{Hom}_{\mathcal{A}}(C, Y) \longrightarrow 0$$

is an epimorphism of abelian groups. This means that $\bar{f}$ is surjective, ensuring that every homomorphism from C to Y factors through f. We know from [2, Proposition 4] that λ-pure quotients in $\mathcal{A}$ are epimorphisms. Recall that a morphism $f : X \to Y$ is an epimorphism if for each pair of morphisms $g, h : Y \to Z$, $gf = hf$ implies $g = h$. Furthermore, by the same method used in the proof of [2, Proposition 3], one can deduce that any λ-pure quotient in $\mathcal{A}$ is a λ-directed colimit of split epimorphisms. This indicates that λ-pure quotients can be constructed as colimits of simpler, well-understood morphisms. In the case where $\lambda = \aleph_0$, $\aleph_0$-pure quotients are nothing other than pure quotients, which are fundamental in the study of pure homological algebra.

In the following, we provide some well-known examples of locally presentable additive categories.

Example 1.2. Let R be an associative ring with $1 \neq 0$ and $(X, \mathcal{O}_X)$ be a scheme.

(i) The category of all left R-modules is locally finitely presentable. By the Lazard-Govorov Theorem, the category of all flat left R-modules is a finitely accessible additive category, which is not locally presentable. Recall that a category $\mathcal{G}$ is said to be finitely accessible if is closed under directed colimits and has a set S of finitely presentable objects such that any object in $\mathcal{G}$ is a directed colimit of objects in S.

(ii) Any Grothendieck category is locally α-presentable for some infinite regular cardinal α.

(iii) If X is quasi-compact and quasi-separated, then by [19], the category $\mathfrak{Qco}X$ of all quasi-coherent sheaves of $\mathcal{O}_X$-modules is locally finitely presentable.

(iv) Let $\mathcal{Q}$ be a quiver (a directed graph) and $\mathrm{Rep}(\mathcal{Q},\mathcal{A})$ be the category of all $\mathcal{A}$-representations of $\mathcal{Q}$. Then, by [1, Corollary 1.54], $\mathrm{Rep}(\mathcal{Q},\mathcal{A})$ is a locally λ-presentable additive category. If $\mathcal{A}$ is locally finitely presentable, then so is $\mathrm{Rep}(\mathcal{Q},\mathcal{A})$.

This paper is organized as follows. Section 2 is devoted to the kernel-cokernel pairs in $\mathcal{A}$. The results are applied in Section 3 where we prove that a kernel-cokernel pair

$$\mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z}$$

is λ-pure if and only if it is a λ-directed colimit of split sequences. Furthermore, we will show that any λ-directed diagram of objects in $\mathcal{A}$ induces a canonical λ-pure kernel-cokernel pair.

2. ON KERNEL-COKERNEL PAIRS

In this section, we will show that $\mathcal{A}$ is a pre-abelian category, meaning that it possesses all kernels and cokernels. Specifically, as a generalization of [2, Proposition 5], we will prove that a morphism in $\mathcal{A}$ is λ-pure monomorphism if and only if its cokernel is a λ-pure epimorphism, and a morphism in $\mathcal{A}$ is a λ-pure epimorphism if and only if its kernel is a λ-pure monomorphism. Consequently, we deduce that any λ-pure monomorphism (resp. epimorphism) in $\mathcal{A}$ induces a λ-pure kernel-cokernel pair. Recall that a *kernel-cokernel pair* (i,p) in a category $\mathcal{G}$ is a pair of composable morphisms $\mathcal{X}' \xrightarrow{i} \mathcal{X} \xrightarrow{p} \mathcal{X}''$ such that i is the kernel of p and p is the cokernel of i. This helps us to introduce an exact structure on $\mathcal{A}$. To illustrate it, we start with a simple lemma.

Lemma 2.1. *Let $\mathcal{G}$ be an additive category with pullbacks and pushouts and $f : X \longrightarrow Y$ be a morphism in $\mathcal{G}$, then*

(i) *f has both a kernel and a cokernel in $\mathcal{G}$.*

(ii) *Let $i : \mathrm{Ker}f \longrightarrow X$ (resp. $p : Y \longrightarrow \mathrm{Coker}f$) be the kernel (resp. cokernel) of f. Then, i (resp. p) is a monomorphism (resp. epimorphism).*

(iii) *Let $i : \mathrm{Ker}f \longrightarrow X$ be the kernel of f. Then, f is a monomorphism if and only if $\mathrm{Ker}f = 0$.*

(iv) *Let $p : Y \longrightarrow \mathrm{Coker}f$ be the cokernel of f. Then, f is an epimorphism if and only if $\mathrm{Coker}f = 0$.*

Proof. (i) The cokernel of f is defined by the following pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & \mathrm{Coker} f \end{array}$$

and the kernel of f is defined by the following pullback diagram

$$\begin{array}{ccc} \mathrm{Ker} f & \longrightarrow & 0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

(ii) Assume that for a given pair $g, h : M \longrightarrow \mathrm{Ker} f$ of morphisms in $\mathcal{G}$, we have $ig = ih$. Therefore, $i(g-h) = 0 = i0$ and hence, by the universal property of the pullback, we have $g - h = 0$. This shows that i is a monomorphism. By a dual argument, we deduce that p is an epimorphism.

(iii) Assume that f is a monomorphism. Then, $fi = 0$ implies that $i = 0$. However, by (ii), i is a monomorphism and hence $\mathrm{Ker} f = 0$. Conversely, assume that $\mathrm{Ker} f = 0$ and $fg = 0$ for some $g : M \longrightarrow X$. So, we have a unique morphism $h : M \longrightarrow 0$ such that $g = 0h = 0$. This shows that f is a monomorphism.

(iv) The proof is dual to (iii).

□

The above lemma allows us to explore further properties of the morphism $f : X \longrightarrow Y$ in the following remark.

Remark 2.2. Let $\mathcal{G}$ be an additive category with pullbacks and pushouts and $f : X \longrightarrow Y$ be a morphism in $\mathcal{G}$, then:

(i) $i : \mathrm{Ker} f \longrightarrow X$ is the kernel of f if and only if for any object $A \in \mathcal{G}$, we have the following exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Ker} f) \xrightarrow{\bar{i}} \mathrm{Hom}_{\mathcal{G}}(A, X) \xrightarrow{\bar{f}} \mathrm{Hom}_{\mathcal{G}}(A, Y) \ .$$

(ii) $p : Y \longrightarrow \mathrm{Coker} f$ is the cokernel of f if and only if for any object $A \in \mathcal{G}$, we have the following exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{G}}(\mathrm{Coker} f, A) \xrightarrow{\bar{p}} \mathrm{Hom}_{\mathcal{G}}(Y, A) \xrightarrow{\bar{f}} \mathrm{Hom}_{\mathcal{G}}(X, A) \ .$$

Recall that an additive category is *pre-abelian* if any morphism has both a kernel and a cokernel. Equivalently, a category is pre-abelian if it is a pre-additive category with all finite limits and finite colimits. The following corollary is a direct consequence of Lemma 2.1.

Corollary 2.3. *Any additive category with pullbacks and pushouts is pre-abelian.*

This corollary shows that any locally presentable additive category is pre-abelian. Now, as an another consequence of Lemma 2.1, we obtain the following result.

Lemma 2.4. *Assume that X and Y be λ-presentable objects in $\mathcal{A}$. Then, for any morphism $f : X \longrightarrow Y$, the cokernel of f is also λ-presentable.*

Proof. The proof is a direct consequence of Lemma 2.1 (i) and [1, Proposition 1.16]. □

Lemma 2.5. *Let $\mathcal{G}$ be an additive category with pushouts and $f : X \longrightarrow Y$ be a split monomorphism. If $j : Y \longrightarrow C$ is the cokernel of f, then, the sequence $X \xrightarrow{f} Y \xrightarrow{j} C$ is a split kernel-cokernel pair in $\mathcal{G}$.*

Proof. By assumption, there exists a morphism $g : Y \longrightarrow X$ such that $gf = \mathrm{id}_X$. Thus, we have the following sequence

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \xrightarrow{j} C$$

in $\mathcal{G}$ such that

$$(\mathrm{id}_Y - fg)f = f - fgf = f - f = 0.$$

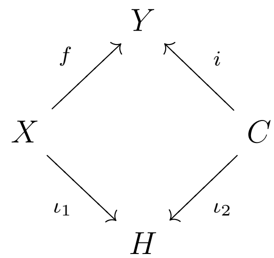
Hence, by definition, there exists a morphism $i : C \longrightarrow Y$ such that $ij = \mathrm{id}_Y - fg$. This shows that

$$jij = (\mathrm{id}_Y - fg) = j - jfg = j - 0 = j.$$

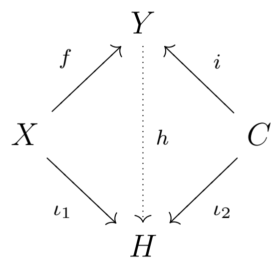
However, by Lemma 2.1 (iii), j is an epimorphism. Therefore, we have $ji = \mathrm{id}_C$ and so, there exists the following split diagram

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{i}{\overset{j}{\rightleftarrows}} C.$$

Consequently, for any commutative diagram



the morphism $h = \iota_1 g + \iota_2 j$ leads to the following commutative diagram



Therfore, Y is isomorphic to $X \bigoplus C$ where

$$X \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} X \bigoplus C \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} C.$$

is both product and coproduct. It follows that $\mathrm{id}_Y = ij + fg$. Now, we will show that $f : X \longrightarrow Y$ is the kernel of j. To this end assume that $k : G \longrightarrow Y$ is a morphism in $\mathcal{A}$ such that $jk = 0$. Then,

$$fgk = (\mathrm{id}_Y - ij)k = k - ijk = k - 0 = k.$$

This show that k factors through f and so, f is the kernel of j. Therefore,

$$X \xrightarrow{f} Y \xrightarrow{j} C$$

forms a split kernel-cokernel pair in $\mathcal{G}$.

□

The proof of the next lemma is closely dual to the proof of the previous lemma.

Lemma 2.6. *Let $\mathcal{G}$ be an additive category with pullbacks and $f : X \longrightarrow Y$ be a split epimorphism. If $i : K \longrightarrow X$ is the kernel of f, then, the sequence $K \xrightarrow{i} X \xrightarrow{f} Y$ is a split kernel-cokernel pair in $\mathcal{G}$.*

These findings indicate that any split monomorphism (respectively, split epimorphism) in $\mathcal{A}$ induces a split kernel-cokernel pair. As a result, we arrive at the following conclusion.

Corollary 2.7. *Let* $\mathcal{E} : X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a kernel-cokernel pair in* $\mathcal{A}$*. Then, we have the following equivalent conditions.*

(i) f *is a split monomorphism*

(ii) g *is a split epimorphism.*

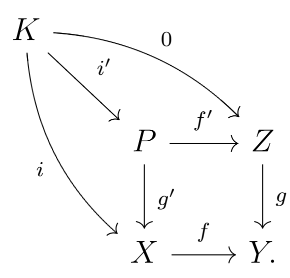
(ii) $\mathcal{E}$ *is split.*

Lemma 2.8. *Let* $\mathcal{G}$ *be an additive category with pullbacks and* $f : X \longrightarrow Y$ *be a morphism in* $\mathcal{G}$*. Then, for any morphism* $g : Z \longrightarrow Y$ *in* $\mathcal{G}$*, we have the following commutative pullback diagram*

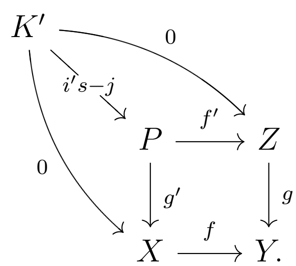
$$\begin{array}{ccccc} K & \xrightarrow{i'} & P & \xrightarrow{f'} & Z \\ \| & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{i} & X & \xrightarrow{f} & Y \end{array}$$

such that i *is the kernel of* f *and* i' *is the kernel of* f'*.*

Proof. Let $i : K \longrightarrow X$ be the kernel of f and $g : Z \longrightarrow Y$ be an arbitrary morphism. By using the universal property of pullback, we obtain the following commutative diagram



Assume that $j : K' \longrightarrow P$ is a morphism in $\mathcal{G}$ such that $f'j = 0$. Therefore, $fg'j = 0$, and hence there exists a morphism $s : K' \longrightarrow K$ such that $is = g'j$. Consequently, $g'i's = g'j$, i.e., $g'(i's - j) = 0$. Thus, we have the following commutative diagram



By the universal property of pullback, we have $i's - j = 0$ and so $i's = j$. Therefore, $i' : K' \longrightarrow P$ is the kernel of f' and hence we have the following commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{i'} & P & \xrightarrow{f'} & Z \\ \| & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{i} & X & \xrightarrow{f} & Y. \end{array}$$

$\square$

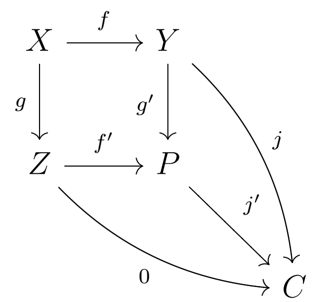
The proof of the next lemma is duall to the proof of the previous lemma. However, we provide a detailed proof for the sake of clarity.

Lemma 2.9. *Let $\mathcal{G}$ be an additive category with pushouts and $f : X \longrightarrow Y$ be a morphism in $\mathcal{G}$. Then, for any morphism $g : X \longrightarrow Z$ in $\mathcal{G}$, we have the following commutative diagram*

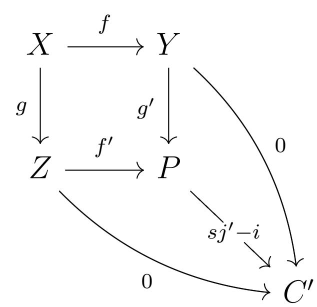
$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{j} & C \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \| \\ Z & \xrightarrow{f'} & P & \xrightarrow{j'} & C \end{array}$$

such that j is the cokernel of f and j' is the cokernel of f'.

Proof. Let $j : Y \longrightarrow C$ be the cokernel of f. For the given morphism $g : X \longrightarrow Z$, we use the universal property of pushout and deduce the following commutative diagram



such that $j'f' = 0$. Assume that $i : P \longrightarrow C'$ is a morphism such that $if' = 0$. Therefore, $ig'f = 0$ and hence we have a morphism $s : C \longrightarrow C'$ such that $sj = ig'$. Consequently, $sj'g' = ig'$, i.e. $(sj' - i)g' = 0$. Thus, we have the following commutative diagram



By the universal property of pushout, we have $sj' - i = 0$ and so $sj' = i$. Consequently, $j' : P \longrightarrow C'$ is the cokernel of f' and so we are done.

□

Proposition 2.10. *The following conditions are satisfied in $\mathcal{A}$.*

(i) *The cokernel of a λ-pure morphism is a λ-pure quotient.*

(ii) *The kernel of a λ-pure quotient is a λ-pure morphism.*

Proof. By employing the same method used in the proof of [2, Proposition 5], we complete the proof. □

Lemma 2.11. *Let $f : X \longrightarrow Y$ be a λ-pure morphism in $\mathcal{A}$. Then,*

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Coker} f$$

is a λ-directed colimit of split kernel-cokernel pairs.

Proof. The proof follows directly from Lemma 2.5 and [2, Example 2 (c)]. □

Lemma 2.12. *Let $f : X \longrightarrow Y$ be a λ-pure quotient in $\mathcal{A}$. If $i : K \longrightarrow X$ is the kernel of f then, $\mathcal{E}: \ K \xrightarrow{i} X \xrightarrow{f} Y$ is a λ-directed colimit of split kernel-cokernel pairs.*

Proof. It is known that there exists a λ-directed system $\{Y_i, f_{ij}\}_{i\in I}$ of λ-presentable objects in $\mathcal{A}$ such that colim$Y_i = Y$. For each $i \in I$, use Proposition 2.10 and consider the following pullback diagram

$$\begin{array}{ccccc} K & \xrightarrow{\iota_i} & P_i & \xrightarrow{f_i} & Y_i \\ \| & & \downarrow h_i & & \downarrow g_i \\ K & \xrightarrow{\iota} & X & \xrightarrow{f} & Y. \end{array}$$

Then, there exists a morphism $t_i : Y_i \longrightarrow X$ in $\mathcal{A}$ such that $ft_i = g$. Thus, by the universal property of pullback, the top row must split. Therefore, we obtain a λ-directed system $\{\mathcal{E}_i : \ K \xrightarrow{\iota_i} P_i \xrightarrow{f_i} Y_i \ \}_{i\in I}$ of split kernel-cokernel pairs in $\mathcal{A}$ such that colim$\mathcal{E}_i = \mathcal{E}$.

□

Theorem 2.13. *Any λ-pure morphism in $\mathcal{A}$ induces a kernel-cokernel pair.*

Proof. Let $f : X \longrightarrow Y$ be a λ-pure-morphism in $\mathcal{A}$ and $j : Y \longrightarrow C$ be the cokernel of f. We aim to prove that $f : X \longrightarrow Y$ is the kernel of j. To this end, assume that $h : H \longrightarrow Y$ is a morphism in $\mathcal{A}$ with $jh = 0$. We need to show that there exists a morphism $h' : H \longrightarrow X$ where $fh' = h$. It is known that we have a λ-directed system $\{H_i, f_{ij}\}_{i\in I}$ of λ-presentable objects in $\mathcal{A}$ such that colim$H_i = H$. So, for each $i \in I$, we have a morphism $hf_i : H_i \longrightarrow Y$ ($f_i : H_i \longrightarrow H$ is the canonical morphism) where $fhf_i = 0$. Then, by Lemma 2.11, there exists a morphism $k_i : H_i \longrightarrow X$ such that $fk_i = hf_i$. Assume that $i < j$. We know that $hf_jf_{ij} = hf_i$ and then,

$$fk_jf_{ij} = hf_jf_{ij} = hf_i = fk_i.$$

Because, f is a monomorphism, we deduce that $k_jf_{ij} = k_i$. So, k=colimk_i does the job. □

Theorem 2.14. *Any λ-pure quotient in $\mathcal{A}$ induces a kernel-cokernel pair.*

Proof. The proof is dual to the proof of the Theorem 2.13.

□

3. λ-PURE EXACT STRUCTURE

In this section, we show that every locally λ-presentable additive category (λ is an infinite regular cardinal) is an exact category equipped with λ-pure kernel-cokernel pairs. We also show that any λ-pure kernel-cokernel pair in a locally λ-presentable additive category is a λ-directed colimit of split kernel-cokernel pairs. This will show that any λ-directed diagram of objects in $\mathcal{A}$ induces a canonical λ-pure kernel-cokernel pair. Before starting, let us recall the definition of an exact category. Let $\mathcal{G}$ be an additive category. By a *conflation* in $\mathcal{G}$, we mean a kernel-cokernel pair $\mathcal{X}' \overset{i}{\longrightarrow} \mathcal{X} \overset{p}{\longrightarrow} \mathcal{X}''$ in $\mathcal{G}$. The map i (resp. p) is called an *inflation* (resp. *deflation*). Let $\mathcal{E}$ be the class of all conflations in $\mathcal{G}$. The pair $(\mathcal{G}, \mathcal{E})$ is said to be an *exact category* if the following axioms hold.

(i) For any object $G \in \mathcal{G}$, the identity morphism 1_G is both inflation and deflation.
(ii) Deflations (resp. Inflations) are closed under composition.
(iii) The pullback (resp. pushout) of a deflation (resp. inflation) along an arbitrary morphism exists and yields a deflation (resp. inflation).

In the following example, we present some specific exact categories.

Example 3.1. Let R be an associative ring with $1 \neq 0$ and $(X, \mathcal{O}_X)$ be an arbitrary scheme.

(i) The category of all flat (resp. absolutely pure) left R-modules is an exact category.
(ii) Any extension closed subcategory of an abelian category is an exact category.
(iii) By [8] the category of all flat (absolutely pure) quasi-coherent sheaves of $\mathcal{O}_X$-modules is an exact category.
(iv) Let $\mathcal{Q}$ be a quiver. By [9], the category of all flat (absolutely pure) $\mathcal{A}$-representations of $\mathcal{Q}$ is an extension closed subcategory of $\mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ and so it is exact.

Notice that, the results presented in Section 2 significantly simplify the λ-purity theory. Specially, the following conclusion can be deduced.

Theorem 3.2. *Let $\mathcal{E}: \ X \overset{i}{\longrightarrow} Y \overset{f}{\longrightarrow} Z$ be a kernel-cokernel pair in $\mathcal{A}$. Then, we have the following equivalent conditions.*

(i) *i is a λ-pure monomorphism.*
(ii) *f is a λ-pure epimorphism.*
(iii) *$\mathcal{E}$ is a λ-directed colimit of split kernel-cokernel pairs.*

(iv) *For any λ-presentable object F, we have the following exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, Z) \longrightarrow 0\ .$$

Proof. The implications (i)⇒(iii) and (ii)⇒(iii) are direct consequence of Lemma 2.11 and 2.12. To prove the implications (iii)⇒(i), (iii)⇒(ii) and (iii)⇒(v), assume that $\mathcal{E}$ is a λ-directed colimit of split kernel-cokernel pairs $\{\mathcal{E}_i :\ X_i \xrightarrow{f_i} Y_i \xrightarrow{p_i} \mathrm{Coker} f_i\ \}_{i \in I}$, where I is a λ-directed poset. Consequently, for any λ-presentable object F in $\mathcal{A}$, the hom functor $\mathrm{Hom}_{\mathcal{A}}(F,\text{-})$ applied to $\mathcal{E}$ yields $\mathrm{Hom}_{\mathcal{A}}(F, \mathcal{E}) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{A}}(F, \mathcal{E}_i)$, which is a short exact sequence of abelian groups. To prove the implication (iv)⇒(i), consider the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{3.1}$$

with λ-presentable objets X' and Y'. To find a morphism $g : Y' \longrightarrow X$ such that $gf' = s$, we note that the diagram (3.1) completes to the following commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & Y' & \xrightarrow{p'} & \mathrm{Coker} f' \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y & \xrightarrow{p} & \mathrm{Coker} f \end{array} \tag{3.2}$$

in $\mathcal{A}$. So, by Lemma 2.4, $\mathrm{Coker} f$ is λ-presentable and hence there exists a morphism $k : \mathrm{Coker} f' \longrightarrow Y$ such that $pk = i$. Hence $p(t - kp') = 0$ and therefore there is a morphism $g : Y' \longrightarrow X$ such that $fg = t - kp'$. Consequently, we obtain $fgf' = tf' - kp'f' = tf' - 0 = tf' = fs$. Since f is a monomorphism, then $gf' = s$. □

The previous characterization of λ-pure morphisms is very useful in applications. Specifically, it leads us to the following definition.

Definition 3.3. A kernel-cokernel pair $\mathcal{E} : X \longrightarrow Y \longrightarrow Z$ in $\mathcal{A}$ is defined to be λ-*pure* if for any λ-presentable object F in $\mathcal{A}$, the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(F, Z) \longrightarrow 0$$

is exact.

The concept of a λ-pure kernel-cokernel pair, as defined in Definition 3.3, enables us to establish a new exact structure on $\mathcal{A}$, known as the λ-*pure exact structure*. In the following result, we demonstrate that for any infinite regular cardinal α, any locally α-presentable additive category can be regarded as an exact category.

Proposition 3.4. *Let $\mathcal{E}$ be the class of all λ-pure kernel-cokernel pairs in $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{E})$ is an exact category.*

Proof. Assume that conflations in $\mathcal{A}$ are precisely all λ-pure kernel-cokernel pairs, inflations are λ-pure monomorphisms, and deflations are λ-pure epimorphisms. So, we deduce the following statements.

(i) For any object $A \in \mathcal{A}$, the identity morphism 1_A is clearly both an inflation and a deflation.
(ii) Composition of two λ-pure monomorphisms (resp. epimorphisms) are λ-pure monomorphisms (rep. epimorphisms), i.e. deflations (resp. inflations) are closed under composition.
(iii) By [2, Proposition 15], the pullback (resp. pushout) of a deflation (resp. inflation) along an arbitrary morphism exists and yields a deflation (resp. inflation).

□

The λ-pure exact structure on $\mathcal{A}$ has many benefits. For an instance, one can use λ-pure projective resolutions and define the λ-pure extension functor $\mathrm{Pext}^n_{\mathcal{A}}(\text{-},\text{-})_\lambda : \mathcal{A}^{\mathrm{op}} \times A \longrightarrow \mathcal{A}b$ for any $n \geq 0$. This provides a natural framework to study the pure homological properties of objects in $\mathcal{A}$ (see [16, 17]). In the next result, we show that any λ-pure extension closed subcategory of $\mathcal{A}$ is an exact subcategory of $\mathcal{A}$. A subcategory $\mathcal{S}$ of $\mathcal{A}$ is said to be λ-pure extension closed if for any λ-pure kernel-cokernel pair $K' \to K \to K''$ in $\mathcal{A}$ with $K', K'' \in \mathcal{S}$, we have $K \in \mathcal{S}$.

Proposition 3.5. *Any λ-pure extension closed subcategory of $\mathcal{A}$ is an exact category.*

Proof. Let $\mathcal{C}$ be a λ-pure extension closed subcategory of $\mathcal{A}$, and let $\mathcal{B}$ be the class of all conflations in $\mathcal{A}$ where all terms are in $\mathcal{C}$. We prove that $(\mathcal{C}, \mathcal{B})$ forms an exact category in $\mathcal{C}$. It is enough to show that conflations in $\mathcal{B}$ are closed under pullbacks and pushouts. To this end, assume that $f : X \longrightarrow Y$ is an inflation in $\mathcal{C}$ and $g : X \longrightarrow Z$ an arbitrary morphism. By Lemma 2.9, we have the following commutative diagram of conflations in $\mathcal{A}$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{j} & C \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \| \\ Z & \xrightarrow{f'} & P & \xrightarrow{j'} & C \end{array}$$

such that j, j' are cokernels of f and f' respectively and X, Y, C, Z are in $\mathcal{C}$. Then, by assumption, we deduce that $P \in \mathcal{C}$. For the second part, let $f : X \longrightarrow Y$ be a deflation in $\mathcal{C}$ and $g : X \longrightarrow Z$ be an arbitrary morphism in $\mathcal{C}$. By Lemma 2.8, we have the following commutative diagram of conflations in $\mathcal{A}$

$$\begin{array}{ccccc} K & \xrightarrow{i'} & P & \xrightarrow{f'} & Z \\ \| & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ K & \xrightarrow{i} & X & \xrightarrow{f} & Y. \end{array}$$

such that i is the kernel of f, i' are kernels of f' and X, Y, C, Z are in $\mathcal{C}$. Therefore, by assumption, we have $P \in \mathcal{C}$.

□

Finally, we apply Proposition 3.4 to deduce that any λ-directed diagram in $\mathcal{A}$ induces a canonical kernel-cokernel pair.

Theorem 3.6. *Let $\{\mathcal{F}_i, f_{ij} | i, j \in I, i \leq j\}$ be a λ-directed diagram in $\mathcal{A}$, then, the following canonical sequence* $\mathcal{K} \xrightarrow{\iota} \bigoplus_{i\in I} \mathcal{F}_i \xrightarrow{\pi} \underset{\overrightarrow{i\in I}}{\text{colim}} \mathcal{F}_i$ *is a λ-pure kernel-cokernel pair.*

Proof. Let $\{\mathcal{F}_i, f_{ij} | i, j \in I, i \leq j\}$ be a λ-directed diagram in $\mathcal{A}$, then, we have the following canonical sequence $\mathcal{K} \xrightarrow{\iota} \bigoplus_{i\in I} \mathcal{F}_i \xrightarrow{\pi} \underset{\overrightarrow{i\in I}}{\text{colim}} \mathcal{F}_i$ of morphisms in $\mathcal{A}$ where ι is the kernel of π. For a given λ-presentable object P in $\mathcal{A}$, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \bigoplus_{i\in I} \text{Hom}_{\mathcal{A}}(P, \mathcal{F}_i) & \xrightarrow{h} & \underset{\overrightarrow{i\in I}}{\text{colim}} \text{Hom}_{\mathcal{A}}(P, \mathcal{F}_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \text{Hom}_{\mathcal{A}}(P, \mathcal{K}) & \xrightarrow{\bar{\iota}} & \text{Hom}_{\mathcal{A}}(P, \bigoplus_{i\in I} \mathcal{F}_i) & \xrightarrow{\bar{\pi}} & \text{Hom}_{\mathcal{A}}(P, \underset{\overrightarrow{i\in I}}{\text{colim}} \mathcal{F}_i) & \longrightarrow & 0 \end{array}$$

of abelian groups where the top row is an exact sequence. We know that by assumption, $g : \underset{\overrightarrow{i\in I}}{\text{colim}}\text{Hom}_{\mathcal{A}}(P,\mathcal{F}_i) \longrightarrow \text{Hom}_{\mathcal{A}}(P, \underset{\overrightarrow{i\in I}}{\text{colim}}\mathcal{F}_i)$ is an isomorphism of abelian groups. Let $t \in \text{Hom}_{\mathcal{A}}(P, \underset{\overrightarrow{i\in I}}{\text{colim}}\mathcal{F}_i)$. Then, there exists $x \in \underset{\overrightarrow{i\in I}}{\text{colim}}\text{Hom}_{\mathcal{A}}(P,\mathcal{F}_i)$ such that $g(x) = t$. However, there exists $y \in \bigoplus_{i\in I}\text{Hom}_{\mathcal{A}}(P,\mathcal{F}_i)$ such that $h(y) = x$. It follows that $\bar{\pi}k(y) = t$ and hence the bottom row must be exact. Therefore, $\mathcal{K} \xrightarrow{\iota} \bigoplus_{i\in I}\mathcal{F}_i \xrightarrow{\pi} \underset{\overrightarrow{i\in I}}{\text{colim}}\mathcal{F}_i$ is a λ-pure kernel-cokernel pair. □

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