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A NOTE ON G-TYPE RINGS

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Abstract. In this article, we introduce and examine the concept of G-type rings. A ring R is classified as a G-type ring if its total quotient ring, denoted by Q, is generated by a countable number of elements over R as an R-algebra. Formally, $Q = R[S^{-1}]$, where S is a countable set of regular elements in R. We establish that R is G-type if and only if there exists a countable set of regular elements, denoted as S, such that every prime ideal disjoint from S consists solely of zero-divisors. It is shown that whenever a ring T is countably generated over a subring R, as an R-algebra, and T is strongly algebraic over R, then R is G-type if and only if T is G-type.

1. Introduction

The concepts of ring extensions, in particular quotient fields and total quotient rings, play an important role in the context of commutative algebra. Thus, the classification of their structure is one of the most interesting subjects for researchers in this area. For instance, Goldman in [5] and Krull in [8] introduced independently and almost at the same time the concept of G-domains, which characterize the domains whose quotient fields are generated by one element over these domains (for some more results concerning this objects, see [6], [4], [10], [3]and [9]). E. Artin and J. T. Tate in [1], have written a valuable note on finite ring extensions. J. Conway Adams in [2] has given the structure of

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G-rings, i.e., the rings R whose total quotient rings are generated by one regular element over R. O. A. S. Karamzadeh and B. Moslemi in [7] have fully characterized G-type domains, i.e., the domains whose quotient fields are countably generated as an R-algebra. They have also established a nice relationship between G-type domains and countable prime avoidance property (CPA property), see [11, Lemma 1.1 and Theorem 2.1] and [7, Definition 3.5, Theorem 3.8, Corollary 3.9 and Theorem 3.10]. Karamzadeh and Moslemi have also used the concept of G-type domains to give a natural proof of the well-known Hilbert Nullstellensatz theorem in countably infinite-dimensional space, see [7, Theorem 2.12, Corollary 2.13 and Corollary 2.14]. In this article we will introduce a new class of rings, G-type rings say, i.e., the rings R whose total quotient rings are generated by a countable number of regular elements over R. An outline of this paper is as follows: Section 2 deals with some preliminary facts about multiplicatively closed sets (mc-set, for brevity) and some needed facts about the total quotient rings. Section 3, investigates the basic properties of G-type rings. Throughout this discussion all rings are commutative with unity. If S is an mc-set in R, then $0 \notin S$, $1 \in S$ and $R[S^{-1}]$ denotes the ring of fractions of R with respect to S. In particular, if $s \in R$ is a non-nilpotent element and $S = \{1, s, s^2, \ldots, s^n, \ldots\}$, then we put $R[S^{-1}] = R_s$. Based on the above argument we observe that if R is a ring and A is a subset of R consisting of some elements of R whose finite products are nonzero, then the set of all finite products of elements of A (the empty product is taken to be the identity) is an mc-set, which is clearly the smallest mc-set containing A (it is also called the mc-set generated by A). If R and T are rings with $R \subseteq T$, the element $t \in T$ is called an integral element over R precisely when $f(t) = 0$ for some monic polynomial $f(x) \in R[x]$. The set of all integral elements of T over R forms a subring of T that contains R, it is called the integral closure of R. We say that R is integrally closed in T if R is its own integral closure in T. The set of all zero-divisors of R is denoted by $Z(R)$ and $S = R \setminus Z(R)$ is an mc-set, which is called the set of regular elements of R. An ideal I of R that contains at least one element of $S = R \setminus Z(R)$ is called a regular ideal. Considering R as a subring of $R[S^{-1}]$, we say that R is integrally closed, if it is integrally closed in $R[S^{-1}]$.

2. **Rings with countably generated total quotient ring**

We begin with the following preliminary facts.

Lemma 2.1. *Let R be a factorial domain and S be an mc-set in R containing a set A of prime elements of R. Then there is an mc-set T with $T \subset S$, $T \cap A = \emptyset$ and $R[S^{-1}] = R[T^{-1}]$.*

Proof. Let C be the mc-set which is generated by A and $B = \{a^2 : a \in A\}$, put $L = B \cup (S \setminus C)$. Now, let T be the smallest mc-set containing L, then $T \subset S$, $T \cap A = \emptyset$ and $R[S^{-1}] = R[T^{-1}]$. We may also simply take T to be $S \setminus A$ and in fact this is the largest mc-set with the previous properties. □

Remark 2.2. For a general ring R, let S be an mc-set consisting of regular elements in R, then we may discard any finite set of such elements from S in the following sense. Let $\{a_1, a_2, \dots, a_n\} \subset S$, put $L = S \setminus \{a_1, a_2, \dots, a_n\}$ and $T = \{a_1^r a_2^r \dots a_n^r x : r = 2, 4, 6, \dots,$ and x is a finite product of elements of $L\} \cup \{1\}$, then $a_i \notin T$, $i = 1, 2, \dots, n$, but $R[T^{-1}] = R[S^{-1}]$. Therefore, there is always an mc-set $T \subset S$ excluding $a_1, a_2, \dots, a_n$ with the property that $R[T^{-1}] = R[S^{-1}]$. The natural question concerning a discarded set is: how large can it be? To settle the question we give the next definition.

Definition 2.3. Let R be a ring, S be an mc-set consisting of regular elements in R and $s, t \in S$. We say that s is equivalent with t and we write $s \sim t$ if and only if $R_t = R_s$.

Lemma 2.4. *Let R be a ring, S be an mc-set consisting of regular elements in R and $s, t \in S$, then $s \sim t$ if and only if there exist integers m and n such that $s^n \in (t)$ and $t^m \in (s)$.*

Proof. Let $s \sim t$, i.e., $R_t = R_s$ and $1/t \in R_t = R_s$, then $1/t = r/s^n$ for some integer n, so that $s^n = rt \in (t)$. Conversely, let $t^m \in (s)$ and $s^n \in (t)$, for some integers m and n, then $t^m = as$ and $s^n = bt$. Now, if $r/t^k \in R_t$, then $r/t^k = rb^k/t^k b^k = l/s^{nk} \in R_s$. □

Corollary 2.5. *If R is a factorial domain, then $s \sim t$ if and only if s and t have the same primes in their factorizations.*

Remark 2.6. As we have observed, $\sim$ defines an equivalence relation in S so that, $S = \bigcup_{i \in I} [s_i]$, where $\{[s_i] : i \in I\}$ is the set of all equivalence classes and each equivalence class $[s_i] = S_i$ is an mc-set with $R[{S_i}^{-1}] = R_{s_i}$. Now, by the Axiom of Choice there exists a set F such that $F \cap [s_i]$ is a singleton set for all $i \in I$. Accordingly, $R[T^{-1}] = R[S^{-1}]$, where T is the mc-set generated by F. It is clear that if $|I|$ is infinite, then $|T| = |I|$.

Definition 2.7. Let R be a ring, S be the set of all regular elements in R, $Q = R[S^{-1}]$ be the total quotient ring of R and let F be as in Remark 2.6, then the caliber of R, denoted by $C(R)$, is the least cardinal λ such that there exists a subset $E \subseteq F$ with $|E| = \lambda$ and $R[T^{-1}] = Q$, where T is the mc-set generated by E, see also [7, Definition 1.1].

Proposition 2.8. *Let R be a ring and let Q, S and F be as in Definition 2.7, then $C(R)$ is a unique cardinal number, which is independent of the chosen set F.*

Proof. Let F_1 and F_2 be two such sets and $E_1 \subseteq F_1$, $E_2 \subseteq F_2$ be two subsets of the least cardinalities such that $R[S_1^{-1}] = R[S_2^{-1}] = Q$, where S_1 and S_2 are the two mc-sets generated by E_1 and E_2 respectively. Now, let $\sim$ be the equivalence relation in Definition 2.3, and let $S = \bigcup_{i \in I}[s_i]$. Put $F_1 \cap [s_i] = \{a_i\}; F_2 \cap [s_i] = \{b_i\}$; $E_1 = \{a_i : i \in J \subseteq I\}$; $E_2 = \{b_i : i \in J^{'} \subseteq I\}$ and $E_3 = \{b_i : \{b_i\} = F_2 \cap [s_i],\ i \in J\}$. Clearly, $R_{s_i} = R_{a_i} = R_{b_i}$. Let S_3 be the mc-set generated by E_3 that is to say, $R[S_3^{-1}] = Q$. This shows that $|E_1| = |E_3| \geq |E_2|$, similarly we have $|E_1| \leq |E_2|$. Therefore, $|E_1| = |E_2|$ and we are through. □

3. G-TYPE RINGS

Definition 3.1. A ring R is said to be a G-type ring if $C(R) \leq \aleph_0$. Equivalently, R is a G-type ring if there exists a countable mc-set S consisting of regular elements in R with $R[S^{-1}] = Q$, where Q is the total quotient ring of R.

Remark 3.2. If $C(R) < \aleph_0$, then $C(R) = 1$ and R is a G-ring, see [2, G-property]. For example, if R is a G-domain (resp., G-type domain) then its caliber is 1 (resp., $C(R) \leq \aleph_0$), see also [7, Definition 1.2].

Before giving some examples, it goes without saying that one can recast the definition of G-type rings as follows. Let R be a ring and Q be its total quotient ring, then R is a G-type ring if Q is generated by a countable number of regular elements over R. Therefore, Q is a countably generated R-algebra and one can write: $Q = R[a_1/b_1, a_2/b_2, \ldots, a_n/b_n, \ldots] = R[1/b_1, 1/b_2, \ldots, 1/b_n, \ldots]$. Clearly, G-type domains, countable rings and factorial domains with only a countable number of prime elements (up to units) are G-type rings. Hence, if R is a countable ring, then $R[x]$ is also a G-type ring, but $R[x]$ is not a G-ring, see [2, Proposition 3.2]. If $R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n \subseteq \ldots$ is a chain of G-type rings such that the regular elements of R_i are regular in R_{i+1} for all i, then so too is $R = \bigcup_{n=1}^{\infty} R_n$ (note, the total quotient ring of R_i is contained in the total quotient ring of R_{i+1}). The following remark is now immediate.

Remark 3.3. Although, $R[x]$ is not a G-ring but $R[x]$ is a G-type ring if and only if R is countable. In generally, if $X = \{x_i\}_{i\in I}$ is a set of indeterminate over R, then $R[X]$ is a G-type ring if and only if $R \cup X$ is a countable set, see also [7, Theorem 2.1 and Corollary 2.2].

We now present the following proposition, which extends [6, Theorem 19].

Proposition 3.4. *Let R be a ring and Q be its total quotient ring and let S be an mc-set in R consisting of regular elements. Then the following statements are equivalent.*

(1) *Each nonzero regular prime ideal intersects S.*
(2) *Each nonzero regular ideal intersects S.*
(3) $Q = R[S^{-1}]$.

Proof. (1) → (2). Let I be an ideal in R containing a regular element and $I \cap S = \emptyset$, then there is a prime ideal $P \supseteq I$ with $P \cap S = \emptyset$, which is absurd.
(2) → (3). Clearly, $S \subseteq Q$. Let $a/b \in Q$, then $(b) \cap S \neq \emptyset$, i.e., there exist $0 \neq r \in R$ with $b' = br \in S$. Thus, $a/b = ar/br = a'/b' \in R[S^{-1}]$.
(3) → (1). In fact we prove more, i.e., (3) → (2). Let $I \neq (0)$ be an ideal and $a \in I$ be a regular element, then $1/a = b/s, s \in S$, i.e., $s = ab \in I \cap S$. □

The following corollary is now immediate.

Corollary 3.5. *A ring R is a G-type ring if and only if there exists a countable mc-set S consisting of regular elements in R such that every prime ideal which is maximal with respect to having the empty intersection with S consists of zero-divisors.*

Proposition 3.6. *Let R be a G-type ring with the total quotient ring Q and let T be a ring with $R \subseteq T \subseteq Q$. Then T is a G-type ring. In particular, a ring R is a G-type ring if and only if every proper localization of R (i.e., $R \subset R[S^{-1}]$, where S is an mc-set consisting of regular elements in R) is a G-type ring.*

The next definition is needed.

Definition 3.7. Let $R \subseteq T$ be a ring extension. Then T is said to be strongly algebraic (resp., strongly integral) over R, if each regular element of R is regular in T and each element $t \in T$ is a root of a polynomial $f(x) \in R[x]$, whose leading coefficient is regular (resp., the unit) in R and moreover if t is regular in T, then the constant term of $f(x)$ is regular in R, too.

Clearly, if $R \subseteq T$ are domains, then T is algebraic (resp., integral) over R if and only if it is strongly algebraic (resp., strongly integral) over R. We require the following lemma.

Lemma 3.8. *Let $R \subseteq T$ be a ring extension such that T is strongly algebraic over R. Then the total quotient ring of T, L say, is integral over the total quotient ring of R, Q say. Moreover, for each regular element $t \in T$ we have $t^{-1} \in Q[t]$.*

Proof. First, let $t \in T$ be a regular element and let $f(x) \in R[x]$, where $f(x) = a_0 + a_1x + \cdots + a_nx^n$ such that both a_0 and a_n are regular and $f(t) = 0$ (note, T is strongly algebraic over R). Consequently, $t(a_1 + a_2t + \cdots + a_nt^{n-1}) = -a_0$. Now, a_0 being regular in R has an inverse in Q, hence we may multiply the latter equality by $-a_0^{-1}$ to get $t^{-1} \in Q[t]$. Finally, let $x = ut^{-1} \in L$, where $u, t \in T$ with t regular. Since T is strongly algebraic over R, we infer that $Q \subseteq L$ and clearly u and t^{-1} are integral over Q, hence x is integral over Q too and we are done. □

We recall that, if $R \subseteq T$ are domains such that T is integral over R, then it is well-known and easy to prove that R is a field if and only if T is a field, see [6, Theorems 15, 16]. The following statement offers a generalization of this fact.

Corollary 3.9. *Let $R \subseteq T$ be a ring extension such that T is strongly algebraic over R. Then if R is its own total quotient ring, so too is T. Moreover, if T is integral over R and T is its own quotient ring, so too is R.*

Proof. Let R be its own quotient ring and $t \in T$ be a regular element. Now in view of the previous lemma, we have $t^{-1} \in R[t] \subseteq T$, hence T is its own quotient ring too. Conversely, let $r \in R$ be regular. Since T is strongly algebraic over R, we infer that r is regular in T too. Therefore $r^{-1} \in T$ and since T is integral over R, we immediately infer that $r^{-1} \in R[r] \subseteq R$, see also [6, Theorem 15], i.e., $r^{-1} \in R$ and we are done. □

The following fact is the counterpart of [6, Theorem 22], which plays an important role in the context of the G-domains, for G-type rings.

Theorem 3.10. *Let $R \subseteq T$ be rings and suppose that T is strongly algebraic over R and countably generated as a ring over R. Then R is a G-type ring if and only if T is a G-type ring.*

Proof. Let Q and L be the total quotient rings of R and T, respectively. Suppose that R is a G-type ring. Hence, $Q = R[S^{-1}]$, where S is a countable mc-set in R consisting of regular elements. We note that $T[S^{-1}] \subseteq L$ is integral over Q, by Lemma 3.8. In view of the latter lemma, we also have $t^{-1} \in Q[t] \subseteq T[S^{-1}]$ for each regular element $t \in T$. Hence $L \subseteq T[S^{-1}]$, i.e., $L = T[S^{-1}]$ and therefore T is a G-type ring (note, this part of the proof did not require T to be countably generated over R). Conversely, let $T = R[\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots]$ be a G-type ring with $L = T[S^{-1}]$, where $S = \{s_1, s_2, \ldots, s_n, \ldots\}$ consists of regular elements. Clearly, the elements s_i^{-1} and α_i; $i = 1, 2, 3, \ldots$ are integral over Q, by Lemma 3.8, and consequently satisfy in equations with coefficients in R, which leading coefficients are regular in R. Let us put $a_{s_i} s_i^{-m_i} + \cdots = 0$ and also $b_i \alpha_i^{n_i} + \cdots = 0$, for $i = 1, 2, \ldots, k, \ldots$. Now, let R_1 be the ring generated by $a_{s_i}^{-1}$ and b_i^{-1} for all i, over R. Clearly, $R \subseteq R_1 \subseteq Q$. Now, it is manifest that L is generated by the set $S^{-1} \bigcup \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$ over R. Consequently, the latter set generates L over R_1. Now, we note that each element of the set $S^{-1} \bigcup \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\}$ is integral over R_1, i.e., L is integral over R_1. Since for each regular element $t \in T$, t^{-1} is integral over R_1, we infer that $t^{-1} \in R_1[t]$, see [6, Theorem 15]. This implies that whenever r is a regular element of R, then $r^{-1} \in R_1[r] = R_1$ (note, r is also regular in T, for T is strongly algebraic over R). Consequently, $Q \subseteq R_1$ and therefore $Q = R_1$. Hence, Q is countably generated over R, as an algebra, and this completes the proof. □

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