Journal of Advanced Mathematical Modeling
ISSN (print): 2251-8088, ISSN (on-line): 2645-6141
Vol. 14 No. 3(2024), pp. 94-107.
DOI: 10.22055/jamm.2025.47618.2300

ANNIHILATOR OF IDEALS IN $C(X)$

ROSTAM MOHAMADIAN*

Dedicated to Professor Ali Rezaei Aliabad on his 70th birthday

Communicated by: A. Rezaei Aliabad

Abstract. Let I be an ideal of $C(X)$. In this paper we show that $\mathrm{Ann}(I) = O^{\beta X \setminus \theta(I)}$ and $m\mathrm{Ann}(I) = O^{\beta X \setminus int_{\beta X}\theta(I)}$, where $\theta(I) = \bigcap_{f \in I} cl_{\beta X} Z(f)$ and mI is the pure part of I. We also show that $\mathrm{Ann}(\mathrm{Ann}(I)) = O^{int_{\beta X}\theta(I)}$ and $m\mathrm{Ann}(\mathrm{Ann}(I)) = O^{cl_{\beta X} int_{\beta X}\theta(I)}$. Finally, we show that a space X is a ∂-space if and only if every nonregular prime ideal of $C(X)$ is a z-ideal.

1. Introduction

Throughout this paper, all rings are commutative with unity. Let R be a ring and $S \subseteq R$. The ideal generated by S is denoted by $< S >$ and $\mathrm{Ann}(S) = \{r \in R : rs = 0,\ for\ all\ s \in S\}$. For $a \in R$ we use $\mathrm{Ann}(a)$ instead of $\mathrm{Ann}(\{a\})$. An element $a \in R$ is said to be a regular (resp., zerodivisor) element if $\mathrm{Ann}(a) = (0)$ (resp., $\mathrm{Ann}(a) \neq (0)$). An ideal I of a ring R is called dense if $\mathrm{Ann}(I) = (0)$. An ideal I of a ring R is called regular if it contains a regular element otherwise it is called nonregular, for details about nonregular ideals, see [14]. Also the intersection of all maximal (resp., minimal prime) ideals containing a is denoted by M_a (resp., P_a). A nonzero ideal is called essential if it intersects every nonzero ideal nontrivially. Max(R), Spec(R), Jac(R) and rad(R) denote the set of all maximal ideals, the set of all prime ideals, the intersection of all maximal

MSC(2020): 00A09, 54C40.
Keywords: Annihilator, Stone–Čech Compactification, Pure Ideal.
Received: 31 July 2024, Accepted: 28 February 2025.
*Corresponding author.

ideals and the intersection of all prime ideals of R, respectively. If $\mathrm{Jac}(R) = (0)$ (resp., $\mathrm{rad}(R) = (0)$), then we call R a semiprimitive (resp., reduced) ring. All topological spaces are completely regular Hausdorff. Let $C(X)$ (resp., $C^*(X)$) be the ring of (resp., bounded) real valued continuous functions on X. For $f \in C(X)$, the zero-set of f is the set $Z(f) = \{x \in X : f(x) = 0\}$ and $Z(X) = \{Z(f) : f \in C(X)\}$. The set-theoretic complement of $Z(f)$ is denoted by $coz(f)$. It is well-known that an ideal I of $C(X)$ is a z°-ideal (resp., z-ideal) if $int_X Z(f) = int_X Z(g)$ (resp., $Z(f) = Z(g)$), $f \in I$ and $g \in C(X)$ implies that $g \in I$. Clearly, every z°-ideal is a z-ideal. It is also well-known that $f \in C(X)$ is a von Neumann regular element if and only if $int_X Z(f) = Z(f)$. It is also easy to see that $\mathrm{Ann}(f) = (0)$ if and only if $intZ(f) = \emptyset$, for $f \in C(X)$. The set of all regular (resp., zerodivisor) elements of $C(X)$ is denoted by $r(X)$ (resp., $zd(X)$). For an ideal I of $C(X)$, we write $Z[I]$ to designate the family of zero-sets $\{Z(f) : f \in I\}$ and $\mathrm{Min}(I)$ denotes the set of all prime ideals minimal over I. υX is the Hewitt real compactification of X and βX is the Stone–Čech compactification of X. A space X is pseudocompact if and only if $\beta X = \upsilon X$. We say that a subset S of X is C-embedded in X if every function in $C(S)$ can be extended to a function in $C(X)$. The space X is normal if and only if every closed subset is C-embedded, see [20]. Every maximal ideal of $C(X)$ is precisely of the form $M^p = \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$, where $p \in \beta X$, see [20, Theorem 6.5]. The prime ideals containing a given prime ideal form a chain, also every prime ideal is contained in a unique maximal ideal M^p, for a unique $p \in \beta X$; and the intersection of all the prime ideals contained in M^p is the ideal $O^p = \{f \in C(X) : p \in int_{\beta X} cl_{\beta X} Z(f)\}$. For a subset $A \subseteq \beta X$, we define $M^A = \{f \in C(X) : A \subseteq cl_{\beta X} Z(f)\}$, $O^A = \{f \in C(X) : A \subseteq int_{\beta X} cl_{\beta X} Z(f)\}$. In particular, if $A \subseteq X$, we denote M^A (resp., O^A) by M_A (resp., O_A) and if $p \in X$, then M^p and O^p are denoted by M_p and O_p, respectively. A subset A of βX is called a round subset if $O^A = M^A$. For $f \in C(X)$ it is easy see that $M_f = M_{Z(f)}$ and $P_f = O_{int_X Z(f)}$. An interesting result of McKnight states for any ideal I of $C(X)$, $O^{\theta(I)} \subseteq I \subseteq M^{\theta(I)}$ where $\theta(I) = \bigcap_{f \in I} cl_{\beta X} Z(f)$, see [17, Theorem 1.3]. This fact is generalized in [5] for semiprimitive Gelfand rings. Also, in Theorem 3.1 of [10], it is shown that an ideal I of $C(X)$ is essential if and only if $\mathrm{Ann}(I) = (0)$. An ideal I of $C(X)$ is called a pure ideal if for each $f \in I$, there exists $g \in I$ such that $f = fg$. For each ideal I of $C(X)$, let $mI = \{f \in C(X) : f \in fI\}$. It is clear that mI is a pure ideal, which is called the pure part of I. An ideal I of $C(X)$ is pure if and only if $I = mI$.

It is shown in Theorem 2.2 of [1] that $mI = O^{\theta(I)}$. $\overline{I}$ is the closure I in the m-topology. It is shown that $\overline{I} = M^{\theta(I)}$, for details see 7Q in [20]. The largest z-ideal contained in I is denoted by I^z and the smallest z-ideal containing I is denoted by I_z, for details see Proposition 1.2 in [16].

Concerning topological spaces and $C(X)$ the reader is referred to [19] and [20], respectively. For more information about algebraic concepts see [9].

The paper is organized as follows. In Section 2, we characterize the annihilator of an ideal in rings of continuous functions. In Section 3, we characterize the annihilator of the annihilator of an ideal and its pure part in rings of continuous functions.

2. Characterization of Ann(I) in $C(X)$

We start this section with two well-known lemmas in which some connections between βX and $\theta(I)$ are mentioned. For more information, see [2].

Lemma 2.1. *Let $f, g \in C(X)$, $A, B \subseteq \beta X$ and I, J be two ideals of $C(X)$. Then the following statements hold:*

(a) $cl_{\beta X} A = \bigcap_{f \in O^A} cl_{\beta X} Z(f)$.

(b) $cl_{\beta X} A = \beta X$ *if and only if* $O^A = (0)$.

(c) $\theta(O^A) = \theta(M^A) = cl_{\beta X} A$.

(d) $M^A = M^B$ *if and only if* $cl_{\beta X} A = cl_{\beta X} B$.

(e) *If* $(cl_{\beta X} coz(f)) \cap X = (cl_{\beta X} coz(g)) \cap X$, *then* $cl_{\beta X} coz(f) = cl_{\beta X} coz(g)$.

(f) $cl_{\beta X}(\beta X \setminus cl_{\beta X} Z(f)) = cl_{\beta X} coz(f)$.

(g) $int_{\beta X}(Z(f^\beta) \cup \theta(I)) = int_{\beta X}(cl_{\beta X} Z(f) \cup \theta(I))$.

(h) $cl_{\beta X}(\beta X \setminus \theta(I)) = cl_{\beta X} int_{\beta X} cl_{\beta X}(\beta X \setminus \theta(I))$.

(i) *Let I be an ideal of $C(X)$ and $f \in C(X)$. If $\theta(I) \subseteq int_{\beta X} cl_{\beta X} Z(f)$ then $f \in I$. The converse is true if $\theta(I)$ is a round subset of βX.*

(j) *Let $A \subseteq \beta X$ and $p \in \beta X$. If $O^A \subseteq M^p$, then $M^A \subseteq M^p$.*

Proof. We only prove part (j). The proof of other statements is straightforward.
j) Suppose on the contrary, that $M^A \not\subseteq M^p$. Then $p \notin cl_{\beta X} A$. Now by part (b) there exist $f \in C(X)$ such that $p \notin cl_{\beta X} Z(f)$ and $A \subseteq int_{\beta X} cl_{\beta X} Z(f)$. This means that $f \in O^A \setminus M^p$ which is a contradiction. □

Lemma 2.2. *The following statements hold for two ideals I, J of $C(X)$:*

(*a*) $\theta(I) = \theta(mI) = \theta(\sqrt{I}) = \theta(\overline{I})$.

(*b*) $\theta(I + J) = \theta(I) \cap \theta(J)$.

(*c*) $\theta(IJ) = \theta(I \cap J) = \theta(I) \cup \theta(J)$.

(*d*) $\theta(I) = \theta(I^z) = \theta(I_z)$.

Proof. It is enough to note that for every ideal I in $C(X)$ we have $mI \subseteq I^z \subseteq I \subseteq \sqrt{I} \subseteq I_z \subseteq \overline{I}$ and $\theta(O^{\theta(I)}) = \theta(I) = \theta(M^{\theta(I)})$. The proof of other statements is straightforward. □

The next proposition is the main result of this article.

Proposition 2.3. *Let I be an ideal of $C(X)$. Then* $\mathrm{Ann}(I) = O^{\beta X \setminus \theta(I)}$.

Proof. Assume that $g \in \mathrm{Ann}(I)$. Then $coz(g) \subseteq \theta(I)$. Hence, $\beta X \setminus \theta(I) \subseteq int_{\beta X} cl_{\beta X} Z(g)$. This implies that $g \in O^{\beta X \setminus \theta(I)}$. Now suppose that $g \in O^{\beta X \setminus \theta(I)}$. Without loss of generality we can consider $g \in C^*(X)$. Then $\beta X \setminus \theta(I) \subseteq int_{\beta X} cl_{\beta X} Z(g)$. We claim that $coz(g) \subseteq \theta(I)$ and then $coz(g) \subseteq cl_{\beta X} Z(f)$, for any $f \in I$. Otherwise, there are $f \in I$ and $p \in coz(g)$ such that $p \notin cl_{\beta X} Z(f)$. Therefore, $p \in \beta X \setminus \theta(I)$ and hence $p \in int_{\beta X} cl_{\beta X} Z(g)$. Since $g \in C^*(X)$, $int_{\beta X} cl_{\beta X} Z(g) = int_{\beta X} Z(g^{\beta})$ and so $p \in int_{\beta X} Z(g^{\beta})$. This implies that $p \notin cl_{\beta X} coz(g^{\beta})$ and hence $p \notin cl_{\beta X} coz(g)$. Since $cl_{\beta X} coz(g^{\beta}) = cl_{\beta X} coz(g)$. This implies that $p \notin coz(g)$ which is not true. □

The following results is immediate.

Corollary 2.4. *For each ideal I of $C(X)$,* $m\mathrm{Ann}(I) = O^{\beta X \setminus int_{\beta X} \theta(I)}$.

Proof. By Theorem 2.2 of [1] and Proposition 2.3 we can write:

$$m\mathrm{Ann}(I) = O^{\theta(\mathrm{Ann}(I))} = O^{cl_{\beta X}(\beta X \setminus \theta(I))} = O^{\beta X \setminus int_{\beta X} \theta(I)}.$$

□

Corollary 2.5. *The following statements hold for two ideals I, J of $C(X)$ and $f, g \in C(X)$:*

(*a*) *An ideal I is dense in $C(X)$ if and only if $\beta X \setminus \theta(I)$ is dense in βX.*

(*b*) $\mathrm{Ann}(I) = \mathrm{Ann}(J)$ *if and only if* $int_{\beta X} \theta(I) = int_{\beta X} \theta(J)$

(*c*) $\mathrm{Ann}(f) = \mathrm{Ann}(g)$ *if and only if* $int_X Z(f) = int_X Z(g)$.

(*d*) $\mathrm{Ann}(I) = \mathrm{Ann}(mI) = \mathrm{Ann}(\sqrt{I}) = \mathrm{Ann}(I^z) = \mathrm{Ann}(I_z) = \mathrm{Ann}(\overline{I})$.

(e) $\mathrm{Ann}(f) = M_{cl_X coz(f)}$.

To some extent, part (a) of the above result shows the connection between the algebraic dense and the topological dense concepts.

In the next proposition, we state another representations for the annihilator of an ideal in $C(X)$. For similar representations as in part (c), see [13]. We need the following lemma.

Lemma 2.6. *For every $f \in C(X)$, we have $coz(f) = \bigcup_{g \in C(X)} Z(1 - fg)$.*

Proof. It is clear that $Z(1 - fg) \subseteq coz(f)$ for every $g \in C(X)$. Conversely, suppose that $a \in coz(f)$ and consider the constant function $g = \frac{1}{f(a)}$. Thus, obviously we have $a \in Z(1 - fg)$. □

Proposition 2.7. *The following statements hold for an ideal I of $C(X)$:*

(a) $\mathrm{Ann}(I) = M^{\beta X \setminus \theta(I)}$.

(b) $\mathrm{Ann}(I) = M^{\beta X \setminus int_{\beta X} \theta(I)}$.

(c) $\mathrm{Ann}(I) = \{f \in C(X) : Z(1 - fg) \subseteq \bigcap Z[I]$ *for all* $g \in C(X)\}$.

Proof. Part (a) follows from the fact that $\beta X \setminus \theta(I)$ is a round subset of βX and part (b) is clear. For part (c), it is clear that $f \in \mathrm{Ann}(I)$ if and only if $coz(f) \subseteq \bigcap Z[I]$ if and only if, by Lemma 2.6, $\bigcup_{g \in C(X)} Z(1 - fg) \subseteq \bigcap Z[I]$. □

Remark 2.8. a) Let I be an ideal of a semiprimitive ring R, then $\mathrm{Ann}(I) = \bigcap_{I \nsubseteq M \in \mathrm{Max}(R)} M$. This fact is the same as Lemma 4.3 of [8], but it is not proved there. We prove it for convenience of the readers. First suppose that $x \in \mathrm{Ann}(I)$ and M is a maximal ideal such that $I \nsubseteq M$. Since $xI = (0) \subseteq M$, we infer that $x \in M$. Now assume that $x \in \bigcap_{I \nsubseteq M \in \mathrm{Max}(R)} M$ and on the contrary let $x \notin \mathrm{Ann}(I)$. Hence, there is $a \in I$ such that $xa \neq 0$. Since $xa \notin \mathrm{Jac}(R)$, there exists a maximal ideal M such that $xa \notin M$. This means that $I \nsubseteq M$ and $x \notin M$ which is not true. If I is an ideal of $C(X)$, then $\theta(I) = \{p \in \beta X : I \subseteq M^p\}$, see 7O in [20]. Therefore, $I \nsubseteq M^p$ if and only if $p \in \beta X \setminus \theta(I)$. This shows that $\mathrm{Ann}(I) = \bigcap_{I \nsubseteq M^p} M^p = M^{\beta X \setminus \theta(I)}$ which is part (a) of the above proposition.

b) Let I be an ideal of a reduced ring R, then $\mathrm{Ann}(I) = \bigcap_{I \nsubseteq P \in \mathrm{Spec}(R)} P$. This fact is the same as Lemma 2.11 of [6]. The proof is similar to part (a).

According to Propositions 2.3 and 2.7 it is easy to see that $\mathrm{Ann}(M^A) = \mathrm{Ann}(O^A)$, for every subset A of βX. Let $I(X)$ denotes the set of all isolated point of X. The space X is called almost discrete if $I(X)$ is dense and X is said to be almost locally compact if it has a dense locally compact subset, see [12]. First, we will introduce some famous ideals in $C(X)$ and then we will give some results about their essentiality. For details about the below ideals see [20, 22, 24, 23, 12, 21, 26].

$C_F(X) = O^{\beta X \setminus I(X)}$ (resp., $C_\epsilon(X) = M^{\beta X \setminus I(X)}$) is the intersection of all essential (resp., essential maximal) ideals in $C(X)$.

$C_K(X) = O^{\beta X \setminus X}$ (resp., $I_\psi(X) = M^{\beta X \setminus X}$) is the intersection of all free (resp., free maximal) ideals in $C(X)$.

$I_\upsilon(X) = O^{\beta X \setminus \upsilon X}$ (resp., $C_\psi(X) = M^{\beta X \setminus \upsilon X}$) is the intersection of all hyper-real free (resp., hyper-real maximal) ideals in $C(X)$.

Corollary 2.9. *The following statements hold:*

(a) $\mathrm{Ann}(C_F(X)) = (0)$ *if and only if* $\mathrm{Ann}(C_\epsilon(X)) = (0)$ *if and only if* $cl_X I(X) = X$; *if and only if* X *is almost discrete.*

(b) $\mathrm{Ann}(C_K(X)) = (0)$ *if and only if* $\mathrm{Ann}(I_\psi(X)) = (0)$ *if and only if* $cl_{\beta X} int_{\beta X} X = \beta X$; *if and only if* X *is almost locally compact.*

(c) $\mathrm{Ann}(C_\psi(X)) = (0)$ *if and only if* $\mathrm{Ann}(I_\upsilon(X)) = (0)$; *if and only if* $cl_{\beta X} int_{\beta X} \upsilon X = \beta X$.

It is a well-known fact that if I and J are two prime ideals or z-ideals of $C(X)$, then $IJ = I \cap J$. In the next lemma, we show that this fact is also holds for the semiprime ideals in $C(X)$. Recall that if $f, g \in C(X)$, then $\mathrm{Ann}(f) = \mathrm{Ann}(g)$ if and only if $int_X Z(f) = int_X Z(g)$, see Lemma 2.1 in [3].

Lemma 2.10. *The following statements hold:*

(a) *Let* I, J *be two semiprime ideals of* $C(X)$. *Then* $IJ = I \cap J$.

(b) *Every semiprime ideal of* $C(X)$ *is an idempotent ideal.*

(c) *Let* I *be a finitely generated ideal of* $C(X)$. *Then* I *is idempotent if and only if* I *is generated by an idempotent.*

(d) *For any* $f, g \in C(X)$, *we have* $\mathrm{Ann}(f) = (g)$ *if and only if* $int_X Z(f) = cl_X coz(g) = coz(g)$.

Proof. We prove part (d). The proof of other statements is straightforward. First suppose that $\text{Ann}(f) =< g >$ for some $g \in C(X)$. By parts (b) and (c) we conclude that $\text{Ann}(f) =< e >= \text{Ann}(1-e)$, for some idempotent $e \in C(X)$. Therefore, $int_X Z(f) = int_X Z(1-e) = Z(1-e) = coz(e) = coz(g) = cl_X coz(e) = cl_x coz(g)$. Next, we let $int_X Z(f) = cl_X coz(g) = coz(g)$. Then there is an idempotent $e \in C(X)$ such that $< g >=< e >$. Thus, $int_X Z(f) = int_X Z(1-e)$ and hence $\text{Ann}(f) = \text{Ann}(1-e) =< e >=< g >$ and we are done. □

The following result is the same as Theorem 3.1 in [10]. Before that, we need the following lemma, the proof of which is obvious.

Lemma 2.11. *Let X be dense in T and A is a closed subset of T. Then we have $X \cap int_T(A) = int_X(A \cap X)$. Therefore, $int_T(A) = \emptyset$ if and only if $int_X(A \cap X) = \emptyset$.*

Corollary 2.12. $\text{Ann}(I) = (0)$ *if and only if* $int_{\beta X}\theta(I) = \emptyset$ *if and only if* $int_X(\theta(I) \cap X) = \emptyset$ *if and only if* $int_X(\bigcap_{f \in I} Z(f)) = \emptyset$.

∂-spaces are first introduced in [15]. A space X is called a ∂-space if the boundary of any zero-set in X is contained in a zero-set with empty interior. For more details about ∂-spaces and examples of ∂-spaces, see [15].

Proposition 2.13. *The following statements hold:*

(a) *X is a normal space if and only if $M_A + M_B = M_{A \cap B}$, for every two closed subsets A and B of X.*
(b) *Let A, B be two closed subsets of βX. Then $M^{A \cap B} = M^A + M^B$.*

Proof. a) See part (b) of Lemma 2.8 in [4].
b) Because βX is normal, it is clear according to part (a). □

Frontier ideals are first introduced in [18]. An ideal I of a ring R is called a frontier ideal if $I = M_a + \text{Ann}(a)$, for some $a \in R$. A ring R is said to be a boundary ring (briefly, ∂-ring) if every frontier ideal of R is a regular ideal. The following lemma is true for every reduced ring.

Lemma 2.14. *Every frontier ideal of $C(X)$ is an essential ideal.*

Proof. Let I be a frontier ideal of $C(X)$. Hence, there exists $f \in C^*(X)$ such that $I = M_{Z(f)} + M_{cl_X coz(f)}$ by part (d) of Corollary 2.5. Therefore, by part (b) of Lemma 2.2 we have $\theta(I) = \partial Z(f) \subseteq \partial Z(f^\beta)$. Hence, $int_{\beta X}\theta(I) \subseteq int_{\beta X}\partial Z(f^\beta) = \emptyset$. This implies that $\text{Ann}(I) = (0)$ by Corollary 2.12. □

Now, according to the results of this article, we combine the two Theorems 2.9 of [18] and 4.4 of [15] regarding to ∂-spaces with a different and shorter proof in the ring $C(X)$. In [18], the proof is stated for the boundary frames. In addition, part (d) of the following theorem is a new characteristic for ∂-spaces. Recall that $\text{Ann}(f) = \text{Ann}(g)$ if and only if $int_X Z(f) = int_X Z(g)$, for $f, g \in C(X)$, see Proposition 1.1 in [15].

The following lemma is Corollary 4.6 in [15].

Lemma 2.15. *X is a ∂-space if and only if βX is so.*

Proposition 2.16. *The following statements are equivalent:*

(a) *X is a ∂-space.*

(b) *$C(X)$ is a ∂-ring.*

(c) *Every nonregular prime ideal of $C(X)$ is a z°-ideal.*

(d) *Every nonregular prime ideal of $C(X)$ is a z-ideal.*

Proof. ($a \Rightarrow b$) According to Lemma 2.15, we may assume that X is compact. Let I be a frontier ideal of $C(X)$. Hence, there exists $f \in C(X)$ such that $I = M_{Z(f)} + M_{cl_X coz(f)}$. Since X is normal, by Proposition 2.13, we conclude that $I = M_{\partial Z(f)} = M_{Z(f)\cap cl_X coz(f)}$. By hypothesis, there is $g \in C(X)$ such that $\partial Z(f) \subseteq Z(g)$ and $int_X Z(g) = \emptyset$. This means that $g \in M_{\partial Z(f)} = I$, i.e., I is regular.

($b \Rightarrow a$) Let $f \in C(X)$. Take $I = M_{Z(f)} + M_{cl_X coz(f)}$. By hypothesis, there is $g \in I$ such that $int_X Z(g) = \emptyset$. Since $I \subseteq M_{\partial Z(f)}$ we infer that $g \in M_{\partial Z(f)}$ and so $\partial Z(f) \subseteq Z(g)$.

($b \Rightarrow c$) Let P be a nonregular prime ideal. We are to show that P is a z°-ideal. Suppose that $\text{Ann}(f) = \text{Ann}(g)$, $f \in P$ and $g \in C(X)$. Put $I = M_f + \text{Ann}(f)$. By hypothesis, $I \cap r(X) \neq \emptyset$. Hence, there is $h \in I$ such that $int_X Z(h) = \emptyset$. Therefore, there are $a \in M_f$ and $b \in \text{Ann}(f)$ such that $h = a + b$. Since $coz(b) \subseteq Z(f) \subseteq Z(a)$ we infer that $ab = 0$ and hence $Z(b^2 + f^2) \subseteq Z(a^2 + b^2) = Z(h)$. On the other hand, $bg = 0 \in P$. If $b \in P$ then $b^2 + f^2 \in P$ which is not true, for $int_X Z(b^2 + f^2) = \emptyset$ and P is nonregular. This implies that $g \in P$ and we are done.

$(c \Rightarrow d)$ It is obvious.

$(d \Rightarrow b)$ Let $I = M_f + \mathrm{Ann}(f)$, for some $f \in C(X)$ and on the contrary suppose that $I \subseteq zd(X)$. Then there is $P \in \mathrm{Min}(I)$ such that $P \subseteq zd(X)$. Since $< f, \mathrm{Ann}(f) > \subseteq P$, then we deduce that P is not a minimal prime ideal. Thus, there is a minimal prime ideal Q of $C(X)$ such that $Q \subsetneqq P$. Suppose that $g \in P \setminus Q$. Also assume that P^* be the largest prime ideal which $g \notin P^*$ and $Q \subseteq P^*$. Since P^* is a lower ideal we infer that it is not a z-ideal, see 14D.4 in [15]. Now P and P^* both containing Q, then must be comparable. But $P \nsubseteq P^*$, for $g \in P \setminus P^*$ and $P^* \subseteq P$ implies that $P^* \subseteq zd(X)$ and by hypothesis P^* should be a z-ideal which is not true. □

A space X is said to be cozero complemented if for each $f \in C(X)$, there is $g \in C(X)$ such that $g \in \mathrm{Ann}(f)$ and $f^2 + g^2 \in r(X)$. Clearly, $f^2 + g^2 \in M_f + \mathrm{Ann}(f)$ and this shows that every cozero complemented space is a ∂-space. To see the generalization of this fact about f-rings, refer to Proposition 3.1 of [18].

A space X is said to be extremally disconnected if every open set has an open closure. It is well-known that X is extremally disconnected if and only if βX is extremally disconnected if and only if for every two closed subsets A, B of βX the equality $int_{\beta X}(A \cup B) = int_{\beta X} A \cup int_{\beta X} B$ holds, see [20], [19] and [2].

Proposition 2.17. *The following statements are equivalent:*

- (a) *X is an extremally disconnected space.*
- (b) *For every two ideals I, J of $C(X)$, $M^{\beta X \setminus \theta(I)} + M^{\beta X \setminus \theta(J)} = M^{\beta X \setminus \theta(I \cap J)}$.*
- (c) *For every two ideals I, J of $C(X)$, $\mathrm{Ann}(I) + \mathrm{Ann}(J) = \mathrm{Ann}(I \cap J)$.*

Proof. $(a \Rightarrow b)$ Let $f \in M^{\beta X \setminus \theta(I \cap J)}$. Then $cl_{\beta X}(\beta X \setminus \theta(I)) \cap cl_{\beta X}(\beta X \setminus \theta(J)) \subseteq cl_{\beta X} Z(f)$. By hypothesis, there are two idempotent $g^\beta, h^\beta \in C(\beta X)$ such that $cl_{\beta X}(\beta X \setminus \theta(I)) = Z(g^\beta)$ and $cl_{\beta X}(\beta X \setminus \theta(J)) = Z(h^\beta)$. Since $Z((g^\beta)^2 + (h^\beta)^2) \subseteq int_{\beta X} Z(f^\beta)$ then by 1D.1 of [20] we infer that f^β is a multiple of $(g^\beta)^2 + (h^\beta)^2$, that is $f^\beta = ((g^\beta)^2 + (h^\beta)^2) k^\beta$ for some $k^\beta \in C(\beta X)$. This implies that $f = (g^2 + h^2)k$. On the other hand, $\beta X \setminus \theta(I) \subseteq int_{\beta X} Z(g^\beta) = int_{\beta X} cl_{\beta X} Z(g) \subseteq cl_{\beta X} Z(g)$ implies that $g \in M^{\beta X \setminus \theta(I)}$. Similarly $h \in M^{\beta X \setminus \theta(J)}$. Then we conclude that $f \in M^{\beta X \setminus \theta(I)} + M^{\beta X \setminus \theta(J)}$. This completes the proof.

$(b \Rightarrow a)$ Assume that A, B be two closed subsets of βX. Put $I = O^A$ and $J = O^B$. Then $\theta(I) = A$ and $\theta(J) = B$. Therefore, $\theta(M^{\beta X \setminus \theta(I \cap J)}) = \theta(M^{\beta X \setminus (A \cup B)}) = \theta(M^{\beta X \setminus A}) \cap$

$\theta(M^{\beta X\setminus B})$. This implies that $int_{\beta X}(A\cup B)=int_{\beta X}A\cup int_{\beta X}B$. This means that βX is an extremally disconnected space and hence X is an extremally disconnected space.
$(a\Leftrightarrow c)$ See Theorems 2.3.2 and 5.3.7 in [2] and Corollary 2.13 in [25]. □

A space X is called basically disconnected if every open cozero-set has an open closure. It is well-known that X is basically disconnected if and only if βX is basically disconnected if and only if for every zero-set Z in βX and every closed subset A of βX the equality $int_{\beta X}(Z\cup A)=int_{\beta X}Z\cup int_{\beta X}A$ holds. For more details, see [20] and [2].

Proposition 2.18. *The following statements are equivalent:*

(a) *X is a basically disconnected space.*

(b) *For every $f\in C(X)$ and every ideal I of $C(X)$, $M^{\beta X\setminus\theta(I)}+M^{\beta X\setminus cl_{\beta X}Z(f)}=M^{\beta X\setminus(cl_{\beta X}Z(f)\cup\theta(I))}$.*

(c) *For every $f\in C(X)$ and every ideal I of $C(X)$, $\mathrm{Ann}(f)+\mathrm{Ann}(I)=\mathrm{Ann}(fI)$.*

Proof. The proof is similar to Proposition 2.17. □

A space X is said to be an F-space if every finitely generated ideal in $C(X)$ is principal. It is well-known that X is an F-space if and only if βX is an F-space if and only if for every two zero-sets Z_1,Z_2 in βX the equality $int_{\beta X}(Z_1\cup Z_2)=int_{\beta X}Z_1\cup int_{\beta X}Z_2$ holds. For more information about F-spaces, see [20] and [2].

Proposition 2.19. *The following statements are equivalent:*

(a) *X is an F-space.*

(b) *For every $f,g\in C(X)$ with $fg=0$, $M_{cl_X coz(f)}+M_{cl_X coz(g)}=C(X)$*

(c) *For every two $f,g\in C(X)$,*

$$M^{\beta X\setminus cl_{\beta X}Z(f)}+M^{\beta X\setminus cl_{\beta X}Z(g)}=M^{\beta X\setminus(cl_{\beta X}(Z(f)\cup Z(g)))}.$$

(d) *For every $f,g\in C(X)$, $\mathrm{Ann}(f)+\mathrm{Ann}(g)=\mathrm{Ann}(fg)$.*

Proof. The proof is similar to that of Proposition 2.17. □

3. Characterization of $\mathrm{Ann}(\mathrm{Ann}(I))$ in $C(X)$

In this section, we characterize $\mathrm{Ann}(\mathrm{Ann}(I))$ and its pure part for an ideal I of $C(X)$ topologically. Recall that $\bigcap Z[\mathrm{Ann}(I)]=\bigcap_{cozI\subseteq Z(f)}Z(f)$, where $cozI=\bigcup_{g\in I}coz(g)$.

Proposition 3.1. *The following statements hold for an ideal I of $C(X)$:*

(a) $\mathrm{Ann}(\mathrm{Ann}(I)) = O^{int_{\beta X}\theta(I)}$.
(b) $\mathrm{Ann}(\mathrm{Ann}(I)) = M^{int_{\beta X}\theta(I)}$.
(c) $\mathrm{Ann}(\mathrm{Ann}(I)) = M^{cl_{\beta X}int_{\beta X}\theta(I)}$.

Proof. We prove part (b). The proof of the other statements are straightforward. By Proposition 2.7, $\theta(\mathrm{Ann}(I)) = \theta(M^{\beta X\setminus\theta(I)}) = cl_{\beta X}(\beta X \setminus \theta(I))$. Therefore, $\mathrm{Ann}(\mathrm{Ann}(I)) = M^{\beta X\setminus\theta(\mathrm{Ann} I)} = M^{\beta X\setminus cl_{\beta X}(\beta X\setminus\theta(I))} = M^{int_{\beta X}\theta(I)}$. □

Proposition 3.2. *Let X be a pseudocompcat space. For each ideal I of $C(X)$ the following statements hold:*

(a) $\overline{I} + \mathrm{Ann}(I) = M^{\partial\theta(I)}$.
(b) $\mathrm{Ann}(\mathrm{Ann}(I)) + \mathrm{Ann}(I) = M^{\partial(int_{\beta X}\theta(I))}$.
(c) $\mathrm{Ann}(\mathrm{Ann}(I)) + \mathrm{Ann}(I) = M^{\partial(cl_{\beta X}int_{\beta X}\theta(I))}$.
(d) $\mathrm{Ann}(M^{\partial\theta(I)}) = \mathrm{Ann}(M^{\partial(int_{\beta X}\theta(I))}) = \mathrm{Ann}(M^{\partial(cl_{\beta X}int_{\beta X}\theta(I))}) = (0)$.

Proof. We only prove part (a). The proof of other statements is straightforward.
$M^{\partial\theta(I)} = M^{\theta(I)\setminus int_{\beta X}\theta(I)} = M^{\theta(I)\cap(\beta X\setminus int_{\beta X}\theta(I))} = M^{\theta(I)} + M^{\beta X\setminus int_{\beta X}\theta(I)}$. This implies that $M^{\partial\theta(I)} = \overline{I} + \mathrm{Ann}(I)$. □

It is well known that if R is reduced ring and $a \in R$, then $P_a = \mathrm{Ann}(\mathrm{Ann}(a))$, see Proposition 1.4 of [7]. Using this and topological representation of M_a, the following result is evident.

Lemma 3.3. *For each $f \in C(X)$, we have $P_f = O^{int_{\beta X}cl_{\beta X}Z(f)}$ and $M_f = M^{cl_{\beta X}Z(f)}$.*

In general, the equality $I = J$ does not follow from the equality of $mI = mJ$. In the next proposition, we see that this criterion holds for certain ideals of $C(X)$. Recall that $mI = mJ$ if and only if $\overline{I} = \overline{J}$.

Proposition 3.4. *Let I, J be two ideals of $C(X)$. Then the following statements are equivalent:*

(a) $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.
(b) $\overline{\mathrm{Ann}(I)} = \overline{\mathrm{Ann}(J)}$.
(c) $m\mathrm{Ann}(I) = m\mathrm{Ann}(J)$.

Proof. The implications $(a \Rightarrow b)$ and $(b \Rightarrow c)$ are clear.
$(c \Rightarrow a)$ By Corollary 2.4 we have $O^{\beta X \setminus int_{\beta X}\theta(I)} = O^{\beta X \setminus int_{\beta X}\theta(J)}$. Therefore, $\beta X \setminus int_{\beta X}\theta(I) = \beta X \setminus int_{\beta X}\theta(J)$ and hence $int_{\beta X}\theta(I) = int_{\beta X}\theta(J)$. Now by part (b) of Corollary 2.5 we have $\text{Ann}(I) = \text{Ann}(J)$. □

The following example shows that $(c \Rightarrow a)$ of Proposition 3.4 is not true, in general.

Example 3.5. Let $R = \frac{F[x]}{I}$, where F is a field and $I =< x^3 >$. We consider two ideals in R, namely $\mathcal{I} =< x + I >$ and $\mathcal{J} =< x^2 + I >$. Then, $\text{Ann}(\mathcal{I}) = \mathcal{J}$ and $\text{Ann}(\mathcal{J}) = \mathcal{I}$. Clearly, $m\mathcal{I} = m\mathcal{J} = (0)$, while $\mathcal{I} \neq \mathcal{J}$.

Proposition 3.6. *For each ideal I of $C(X)$, $m\text{Ann}(\text{Ann}(I)) = O^{cl_{\beta X} int_{\beta X}\theta(I)}$.*

Proof. By Corollary 2.4 and Proposition 3.1 it is clear. □

We conclude the paper with the following proposition, which states a topological statement when the ideals mI and $\text{Ann}(I)$, for an ideal I of $C(X)$ are summand.

Proposition 3.7. *For each ideal I of $C(X)$, $\theta(I)$ is open if and only if $mI \oplus \text{Ann}(I) = C(X)$.*

Proof. $(\Rightarrow)$ Since $\theta(I)$ is clopen, then there is $f \in C(X)$ such that $Z(f) = int_X Z(f)$ and $\text{Ann}(I) = \text{Ann}(f)$. Hence, there exists $g \in C(X)$ such that $f = f^2 g$ and also $f \in I$. Therefore, $fg \in mI$ and $1 - fg \in \text{Ann}(I)$. Now $1 = fg + (1 - fg)$ shows that $mI \oplus \text{Ann}(I) = C(X)$.
$(\Leftarrow)$ Suppose on the contrary, that there exists $p \in \theta(I) \setminus int_{\beta X}\theta(I)$. Then $M^{\beta X \setminus \theta(I)} = \text{Ann}(I) \subseteq M^p$ and $mI \subseteq M^P$. This implies that $mI \oplus \text{Ann}(I) \subseteq M^p$ which is a contradiction. □

Acknowledgments

The author would like to thank the referees for reading the article carefully and giving useful comments. Also the author are grateful to the Research Council of Shahid Chamran University of Ahvaz financial support (GN:SCU.MM403.648).

References

[1]] Abu Osba, E., Al-Ezeh, H., 2003. The pure part of the ideals in $C(X)$, *Math. J. Okayama Univ.*, (45), 73–82.

[2] Aliabad, A.R., 1996. *z°-ideals in $C(X)$*, Ph.D. thesis, Department of Mathematics, Shahid Chamran University of Ahvaz, Iran.

[3] Azarpanah, F., Karamzadeh O.A.S. and Rezaei Aliabad, A., 1999. On z°-ideals in $C(X)$, *Fund. Math.*, (160), 15-25. doi:10.4064/fm1999160111525

[4] Aliabad, A.R., Azarpanah, F. and Paimann, M., 2010. z-ideals and z°-ideals in the factor rings of $C(X)$, *Bull. Iran. Math. Soc.*, *Vol. 36* No. (1), pp.211-226.

[5] Aliabad, A.R., Badiei, M. and Nazari, S., 2022. On commutative Gelfand rings, *J. Math. Ext.*, *Vol. 16*, No. 8 (4), 1-22. doi:10.30495/JME.2022.1866

[6] Aliabad, A.R., Hashemi, J. and Mohamadian, R., 2016. P-ideals and PMP-ideals in commutative rings, *J. Math. Ext.*, *Vol. 10*, No. (4), 19-33.

[7] Aliabad, A.R. and . Mohamadian, R, 2011. On sz°-ideals in polynomial rings, *Comm. Algebra*, *39*, 701-717. doi:10.1080/00927871003591884

[8] Aliabad, A.R. and Mohamadian, R. and Nazari, S., 2017. On regular ideals in reduced rings, *Filomat 31* (12), 3715-3726. doi:10.2298/FIL1712715A

[9] Atiyah, M.F., Macdonald, I.G., 1969. *Introduction to Commutative Algebra*, Addison-Wesely, Reading Mass.

[10] Azarpanah, F., 1995. Essential ideals in C(X), *Period. Math. Hungar.*, *31*, 105-112. doi:10.1007/BF01876485

[11] Azarpanah, F., 1997. Intersection of essential ideals in $C(X)$, *Proc. Amer. Math. Soc.*, *125*, 2149-2154. doi:10.1090/S0002-9939-97-04086-0

[12] Azarpanah, F., 1989. Algebraic properties of some compact spaces, *Real Analysis Exchange*, 317-328.

[13] Azarpanah, F., Ghirati, M. and Taherifar, A., 2018. Closed ideals in $C(X)$ with different representations, *Houston J. Math.*, *vol. 44*, No. 1, 363-383.

[14] Azarpanah, F., Karamzadeh, O.A.S. and Rezaei Aliabad, A., 2000. On ideals consisting entirely of zero divisors, *Comm Algebra*, *28*, 1061-1073. doi:10.1080/00927870008826878

[15] Azarpanah, F. and Karavan, M., 2005. On nonregular ideals and z-ideals in $C(X)$, *Czechoslovak Math. J.*, *55*, 397-407. doi:10.1007/s10587-005-0030-0

[16] Azarpanah, F. and Mohamadian, R., 2007. $\sqrt{z}$-ideals and $\sqrt{z^\circ}$-ideals in $C(X)$, *Acta Math. Sinica*, English Series, *23*, 989-996. doi:10.1007/s10114-005-0738-7

[17] Dietrich, W. 1970. On the ideal structure of C(X), *Trans. Amer. Math. Soc.*, *152*, 61-77. doi:10.2307/1995638

[18] Dube, T. and Nsayi, J.N., 2016. Another ring-theoretic characterization of boundary spaces, *Houston J. Math.*, *Vol. 42*, No. 2, 709-722.

[19] Engelking, R., 1977. *General Topology*, Warsaw Poland: PWN-polish Science Publication.

[20] Gillman, L. and Jerison, M., 1976. *Rings of Continuous Functions*, Berlin, Germany: Springer-Verlag.

[21] Ghirati, M. and Taherifar, A., 2014. Intersections of essential (resp., free) maximal ideals of $C(X)$, *Topol. Appl.*, *167*, 62-68. doi:10.1016/j.topol.2014.03.007

[22] Johnson, D.G. and Mandelker, M., 1973. Functions with pseudocompact support, *Gen. Topol. Appl.*, (3), 331-338. doi:10.1016/oo16-660X(73)90020-2

[23] Karamzadeh, O.A.S. and Rostami, M., 1985. On the intrinsic topology and some related ideals of $C(X)$, *Proc. Am. Math. Soc.*, *93*(1), 179–184. doi:10.1090/S0002-9939-1985-0766552-9

[24] Mandelker, M., 1971. Supports of continuous functions, *Trans. Am. Math. Soc.*, *156*, 73–83. doi:10.1090/S0002-9947-1971-0275367-4

[25] Taherifar, A., 2014. Some new classes of topological spaces and annihilator ideals, *Topol. Appl.*, *165*, 84-97. doi:10.1016/j.topol.2014.01.017

[26] Taherifar, A., 2014. Intersections of essential minimal prime ideals, *Comment. Math. Univ. Carol.*, *55*(1), 121-130.

Rostam Mohamadian
Department of Mathematics,
Faculty of Mathematical Sciences and Computer,
Shahid Chamran University of Ahvaz,
Ahvaz, Iran
Email: mohamadian_r@scu.ac.ir
