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BEST PROXIMITY POINT OF WEAK CONTRACTION ON THE RIEMANNIAN MANIFOLDS

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Abstract. This study aims to define ⊥-proximally increasing mappings and compute some best proximity point results regarding this mapping in the framework of new spaces on the Riemannian manifolds. The new spaces are called strongly orthogonal Riemannian metric spaces.

1. Introduction

Fixed point theory has been widely utilized in nonlinear analysis since Banach introduced his renowned contraction principle in 1922. Numerous researchers have expanded this theory in two ways and have applied these extensions for various purposes, such as establishing the existence and uniqueness of solutions for integral, ordinary, and partial differential equations, enhancing iterative algorithms, and addressing engineering challenges. One approach involves the establishment of new contractions to prove the existence of a fixed point for certain mappings that satisfy specific criteria.

In 2011, Raj [19] redefined the concept of the best proximity point and established several theorems for weakly contractive nonself-mappings. This seminal work laid the foundation for the best proximity point theory, which aims to delineate conditions ensuring the existence of these critical points. Consequently, numerous scholars have delved into

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diverse contraction types to corroborate the presence of best proximity points in varied metric spaces and ordered metric spaces, as detailed in the references [3, 22], and their corresponding works. Although the most effective approximation theorems are suitable for providing an approximate solution to the equation $Tx = x$, these results may not yield an optimal approximate solution. Conversely, best proximity point theorems provide an optimal approximate solution. Specifically, a best proximity point theorem outlines the necessary conditions for the existence of an element x that minimizes the error $d(x, Tx)$.

Recently, Eshaghi et al. [6] introduced the concept of orthogonal sets and orthogonal Riemannian metric spaces. They also demonstrated the existence of the Banach fixed point theorem in orthogonal Riemannian metric spaces [6], and further extensions of this theorem have been derived in [2, 17, 16]. Additionally, there have been notable generalizations of the Banach contraction principle.

In 1969, Ky Fan [7] introduced a fixed point problem for non-self mapping, focusing on the concept of the best proximity point. This theory is an extension of the fixed point theorem, with the best proximity point theorem being a natural outcome. Notably, significant best proximity point effects have been observed in [10, 20, 23]. For in-depth understanding and practical implementations, it is advisable to consult references [9, 8]. Recent deliberations have delved into the existence of fixed points and best proximity points for specific mappings in Riemannian metric spaces and orthogonal Riemannian metric spaces.

2. Preliminaries

In this section, we state the main definitions required.

Definition 2.1. Let $\mathcal{M}$ be a smooth manifold. The 2-tensor positive definite field $\mathfrak{g}$ on $\mathcal{M}$ that is covariant, symmetric and smooth is called a Riemannian metric on $\mathcal{M}$. In this case, we call the pair $(\mathcal{M}, \mathfrak{g})$ a Riemannian manifold [12].

This imply that for some smooth coordinate $(\mathcal{U}, \mathfrak{a})$ on l-dimensoinal manifold $\mathcal{M}$, the l^2 functions,

$$\mathfrak{g}(\frac{\partial}{\partial \mathfrak{a}^i}, \frac{\partial}{\partial \mathfrak{a}^j}) : \mathcal{U} \to \mathbb{R},$$

are smooth. The set $\mathfrak{g}_p$ of inner products creates a Riemannian metric. For all point $p \in \mathcal{M}$, a Riemannian metric represent a positive definite inner product $\mathfrak{g}_p : \mathcal{T}_p\mathcal{M} \times \mathcal{T}_p\mathcal{M} \to \mathbb{R}$, along with which comes a norm $|\cdot|_p : \mathcal{T}_p\mathcal{M} \to \mathbb{R}$ determined by $|v|_p = \sqrt{\mathfrak{g}_p(v,v)}$.

Definition 2.2. Assume that $(\mathcal{M}, \mathfrak{g})$ is a Riemannian manifold and $\gamma : [a,b] \to \mathcal{M}$ is a curve segment that is piecewise smooth, the integral

$$L(\gamma) = \int_a^b |\gamma'(t)|_{\gamma(t)}\, dt,$$

is well-defined and is called length of γ. For any differentiable curve that is continuous piecewise, we can determine its length by efficiently expanding this description.

Definition 2.3. Precisely, define $\mathfrak{d}_{\mathfrak{g}} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by

$$\mathfrak{d}_{\mathfrak{g}}(p,q) = \inf\{L(\gamma) : \gamma \text{ a piecewise continuously differentiable curve from } p \text{ to } q\},$$

were satisfies all of the axioms of a Riemannian metric.

Definition 2.4. ([6]) Assume that $\mathcal{M}$ is a Riemannian manifold and $\perp$ be a binary relation represented on $\mathcal{M} \times \mathcal{M}$. If $\perp$ satisfies the subsequent condition.

$$\exists \mathfrak{a}_0;\ ((\forall \mathfrak{b}; \mathfrak{b} \perp \mathfrak{a}_0) \quad or \quad (\forall \mathfrak{b}; \mathfrak{a}_0 \perp \mathfrak{b})),$$

it is called an orthogonal Riemannian manifold (briefly ORM-st). The element $\mathfrak{a}_0$ is called an orthogonal element.

Suppose that $(\mathcal{M}, \perp)$ be an orthogonal set and $\mathfrak{d}$ be a Riemannian metric on $\mathcal{M}$. Then, we say that $(\mathcal{M}, \perp, \mathfrak{d})$ is an orthogonal Riemannian metric space.

Example.([4]) Let $\mathcal{M} = [2, \infty)$, we define $\mathfrak{a} \perp \mathfrak{b}$ if $\mathfrak{a} \leq \mathfrak{b}$ then by putting $\mathfrak{a}_0 = 2$, $(\mathcal{M}, \perp)$ is an ORM-st.

Definition 2.5. ([6]) Let $(\mathcal{M}, \perp)$ be an orthogonal Riemannian manifold (ORM-st). An orthogonal sequence (shortly O-sequence) is a sequence like $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ such that for every n in $\mathbb{N}$, we have

$$\mathfrak{a}_n \perp \mathfrak{a}_{n+1} \quad or \quad \mathfrak{a}_{n+1} \perp \mathfrak{a}_n.$$

Also, a Cauchy sequence $\{\mathfrak{a}_n\}$ is declared a Cauchy O-sequence if for every n in $\mathbb{N}$,

$$\mathfrak{a}_n \perp \mathfrak{a}_{n+1} \quad or \quad \mathfrak{a}_{n+1} \perp \mathfrak{a}_n.$$

Definition 2.6. [6] A mapping $\mathcal{S} : \mathcal{M} \longrightarrow \mathcal{M}$ is said to be orthogonal preserving ($\perp$ preserving) if $\mathfrak{c} \perp \mathfrak{b}$ implies $\mathcal{S}(\mathfrak{c}) \perp \mathcal{S}(\mathfrak{b})$ for all $\mathfrak{c}, \mathfrak{b} \in \mathcal{M}$.

Definition 2.7. [6] An O-complete Riemannian metric space $(\mathcal{M}, \perp, \mathfrak{d})$ is one where every Cauchy O-sequence converges in $\mathcal{M}$.

Remark that every complete Riemannian metric space is O-complete, but the contrary is not true.

Definition 2.8. ([17]) Suppose that $(\mathcal{M}, \perp)$ is an orthogonal set (ORM-st). Then a sequence $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ is said to be strongly orthogonal (shortly SO-sequence) when for all $n, t \in \mathbb{N}$ we have

$$\mathfrak{a}_n \perp \mathfrak{a}_{n+t} \quad or \quad \mathfrak{a}_{n+t} \perp \mathfrak{a}_n.$$

Definition 2.9. ([17]) Let $\mathcal{M}$ be an orthogonal set. Then, $\mathcal{M}$ is SO-complete (strongly orthogonal complete) if every Cauchy SO-sequence converges.

Understanding the concept of complete Riemannian metric space can be interesting, but it is essential to know that it naturally leads to SO-completeness. However, it is important to note that being SO-complete does not necessarily imply complete Riemannian metric space ([17]).

Definition 2.10. ([17]) A mapping $\mathcal{S} : \mathcal{M} \longrightarrow \mathcal{M}$ is said to be strongly orthogonal continuous (SO-continuous) in $\mathfrak{c} \in \mathcal{M}$ if for every SO-sequence $\{\mathfrak{c}_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that $\mathfrak{c}_n \longrightarrow \mathfrak{c}$, it follows that $\mathcal{S}(\mathfrak{c}_n) \longrightarrow \mathcal{S}(\mathfrak{c})$. Moreover, $\mathcal{S}$ is said to be SO-continuous on $\mathcal{M}$ if it is SO-continuous for each $\mathfrak{c} \in \mathcal{M}$.

It is evident that each continuous mapping is SO-continuous. However, the converse is not necessarily valid, as demonstrated in [17].

Example.([17]) Assume that $\mathcal{M} = (0, \infty)$ and $\mathfrak{d}$ is a normal Riemannian metric. Also, we define $\mathfrak{a} \perp \mathfrak{b}$ if $\mathfrak{a}\mathfrak{b} = \mathfrak{a}$. It is not difficult to realize that $(\mathcal{M}, \perp)$ is an ORM-set. Let $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ specified by

$$\mathcal{S}\mathfrak{a} = \begin{cases} \dfrac{1+\mathfrak{a}}{2}, & \mathfrak{a} \leq 1 \\ \dfrac{1}{2}, & \mathfrak{a} > 1. \end{cases}$$

The following can be concluded:

1) $\mathcal{M}$ is not complete but it is SO-complete. This means that if we have an optional Cauchy sequence $\{\mathfrak{a}_n\}$ in $\mathcal{M}$, then there exists a natural number n_0 such that $\mathfrak{a}_n = 1$ for all $n \geq n_0$. Therefore, the sequence $\{\mathfrak{a}_n\}$ is the constant sequence 1 and as a result, $\mathfrak{a}_n$ converges to $\mathfrak{a} = 1$.
2) The statement is that $\mathcal{S}$ is SO-continuous but not continuous. To prove this, let $\{\mathfrak{a}_n\}$ be an SO sequence that converges to a point $\mathfrak{a} \in \mathcal{M}$. Using step 1, we can find a value $n_0 \in \mathbb{N}$ so that $\mathfrak{a}_n = 1$ for all $n \geq n_0$ and $\mathfrak{a} = 1$. This means that $\mathcal{S}(\mathfrak{a}_n) \to 1$ as n approaches infinity, which is equal to $\mathcal{S}(\mathfrak{a})$.
3) $\mathcal{S}$ is orthogonal preserving. Actuality, if $\mathfrak{a} \perp \mathfrak{b}$, then $\mathfrak{b} = 1$. By description of $\mathcal{S}$, we understand that $\mathcal{S}(\mathfrak{b}) = 1$ and $\mathcal{S}(\mathfrak{a})\mathcal{S}(\mathfrak{b}) = \mathcal{S}(\mathfrak{a})$, this implies $\mathcal{S}(\mathfrak{a}) \perp \mathcal{S}(\mathfrak{b})$.

Very recently, many authors continued this extension and examined on the existence of fixed points for contractive mappings under various constraints on orthogonal Riemannian metric spaces in [2, 17, 5, 16, 21] and references contained therein. Now, let $\mathcal{H}$ and $\mathcal{K}$ be two non-empty subsets of a Riemannian metric space $(\mathcal{M}, \mathfrak{d})$ and $\mathcal{S} : \mathcal{H} \to \mathcal{K}$ be a non-self mapping. The point $\mathfrak{a} \in \mathcal{H}$ is the best proximity point of $\mathcal{S}$ if $\mathfrak{d}(\mathfrak{a}, \mathcal{S}\mathfrak{a}) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$, where $\mathfrak{d}(\mathcal{H}, \mathcal{K}) = \inf\{\mathfrak{d}(\mathfrak{a}, \mathfrak{b}); \mathfrak{a} \in \mathcal{H},\ \mathfrak{b} \in \mathcal{K}\}$. The best proximity point theory is designed to provide enough conditions for the existence of such points. This theorem is a natural extension of a fixed point theorem. You can find some interesting best proximity point results in Riemannian metric and partially ordered Riemannian metric spaces in [1, 10, 11, 15, 18, 20, 19, 23]. Also, for applications, one can refer to [9, 8, 13, 14]. For two non-empty subsets A and B, consider sets $\mathcal{H}_0$ and $\mathcal{K}_0$ defined as follows:

$$\mathcal{H}_0 = \{\mathfrak{a} \in \mathcal{H} : \mathfrak{d}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \text{ for some } \mathfrak{b} \in \mathcal{K}\},$$

$$\mathcal{K}_0 = \{\mathfrak{b} \in \mathcal{K} : \mathfrak{d}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \text{ for some } \mathfrak{a} \in \mathcal{H}\}.$$

To identify the adequate preconditions for the non-emptiness of $\mathcal{H}_0$ and $\mathcal{K}_0$, it is recommended to consult reference [10].

Definition 2.11. ([19]) Let $(\mathcal{H}, \mathcal{K})$ denote a pair of non-empty subsets of $(\mathcal{M}, \mathfrak{d})$, where $\mathcal{H}_0$ is non-empty. The pair $(\mathcal{H}, \mathcal{K})$ is stated to exhibit the P-property if and only if

$$\begin{cases} \mathfrak{d}(\mathfrak{a}_1, \mathfrak{b}_1) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \\ \mathfrak{d}(\mathfrak{a}_2, \mathfrak{b}_2) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \end{cases} \Rightarrow \mathfrak{d}(\mathfrak{a}_1, \mathfrak{a}_2) = \mathfrak{d}(\mathfrak{b}_1, \mathfrak{b}_2),$$

where $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{H}_0$ and $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{K}_0$.

It is straightforward to investigate that for a non-empty subsets $\mathcal{H}$ of $(\mathcal{M}, \mathfrak{d})$, the couple $(\mathcal{H}, \mathcal{H})$ has the P-property.

Definition 2.12. [18] Consider a non-empty set $\mathcal{M}$, which is a partially ordered set concerning the relation $\preceq$ and also has a distance function $\mathfrak{d}$. Let $\mathcal{H}$ and $\mathcal{K}$ be two non-empty subsets of $\mathcal{M}$. A function $\mathcal{S} : \mathcal{H} \to \mathcal{K}$ is stated to be proximally increasing if it pleases the requirement that

$$\begin{cases} \mathfrak{b}_1 \preceq \mathfrak{b}_2 \\ \mathfrak{d}(\mathfrak{a}_1, \mathcal{S}\mathfrak{b}_1) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \\ \mathfrak{d}(\mathfrak{a}_2, \mathcal{S}\mathfrak{b}_2) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \end{cases} \Rightarrow \mathfrak{a}_1 \preceq \mathfrak{a}_2,$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{H}$.

The rest of the article is organized as follows: In Section 1, we recall some definition and propositions which are necessary for later section. In Section 2, we examine the existence of the best proximity point for some contractions in strongly orthogonal Riemannian metric spaces.

3. Main Results

To begin, let's establish the following definition with a comprehensive level of detail.

Definition 3.1. A function $\mathcal{S} : \mathcal{H} \longrightarrow \mathcal{K}$v is stated to be $\perp$-proximally increasing if it satisfies the subsequent necessary condition

$$\begin{cases} \mathfrak{b}_1 \perp \mathfrak{b}_2 \\ \mathfrak{d}(\mathfrak{a}_1, \mathcal{S}\mathfrak{b}_1) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \\ \mathfrak{d}(\mathfrak{a}_2, \mathcal{S}\mathfrak{b}_2) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \end{cases} \Rightarrow \mathfrak{a}_1 \perp \mathfrak{a}_2,$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{H}$.

We will now present and prove the existence result of this section.

Theorem 3.2. *Let $(\mathcal{M}, \perp, \mathfrak{d})$ is strongly orthogonal complete Riemannian metric space and $(\mathcal{H}, \mathcal{K})$ be a pair of closed non-empty subset of the space $(\mathcal{M}, \perp, \mathfrak{d})$ with $\mathcal{H}_0 \neq \emptyset$. Let $\mathcal{S} : \mathcal{H} \longrightarrow \mathcal{K}$ be a map which* *i)* $\mathcal{S}$ *is a* $\perp$*-proximally increasing and* $\perp$*-preserving such that* $\mathcal{S}(\mathcal{H}_0) \subseteq \mathcal{K}_0$*,* $(\mathcal{H}, \mathcal{K})$ *satisfies the P-property;*

ii) *The orthogonal elements* $\mathfrak{a}_0$ *and* $\mathfrak{a}_1$ *exist in* $\mathcal{H}_0$ *such that* $\mathfrak{d}(\mathfrak{a}_1, \mathcal{S}\mathfrak{a}_0) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$*;*

iii) $\mathcal{S}$ *is a SO-continuous function on* $\mathcal{H}$ *such that*

$$\mathfrak{d}(\mathcal{S}\mathfrak{a}, \mathcal{S}\mathfrak{b}) \leq \chi(\mathfrak{d}(\mathfrak{a}, \mathfrak{b})), \tag{3.1}$$

for any point $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* $\mathcal{H}$ *such that* $\mathfrak{a} \perp \mathfrak{b}$ *and* $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ *non-decreasing function with* $\lim_{n\to\infty} \chi^n(\theta) = 0$*, for all* $\theta > 0$*.*

Then $\mathcal{S}$ *has a best proximity point* $\mathfrak{a}^* \in \mathcal{H}$ *such that* $\mathfrak{d}(\mathfrak{a}^*, \mathcal{S}\mathfrak{a}^*) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$*.*

Proof. By (ii), there exist $\mathfrak{a}_0$ and $\mathfrak{a}_1$ in $\mathcal{H}_0$ such that $\mathfrak{d}(\mathfrak{a}_1, \mathcal{S}\mathfrak{a}_0) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$ and $\mathfrak{a}_0 \perp \mathfrak{a}_1$. Since $\mathcal{S}\mathfrak{a}_1 \in \mathcal{S}(\mathcal{H}_0) \subseteq \mathcal{K}_0$, there exist element $\mathfrak{a}_2$ in $\mathcal{H}_0$ such that $\mathfrak{d}(\mathfrak{a}_2, \mathcal{S}\mathfrak{a}_1) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$. Since $\mathcal{S}$ is a $\perp$-preserving and $\perp$-proximally increasing, we get $\mathfrak{a}_1 \perp \mathfrak{a}_2$. Continuing this process, we can construct a sequence $\{\mathfrak{a}_n\}$ in $\mathcal{H}_0$ such that

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathcal{S}\mathfrak{a}_n) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \quad , \quad for\ all\ n \in \mathbb{N}, \tag{3.2}$$

$$with \quad \mathfrak{a}_0 \perp \mathfrak{a}_1,\ \mathfrak{a}_1 \perp \mathfrak{a}_2,\ \mathfrak{a}_2 \perp \mathfrak{a}_3,\ \ldots,\ \mathfrak{a}_n \perp \mathfrak{a}_{n+1},\ \ldots.$$

Thus $\{\mathfrak{a}_n\}$ is an O-sequence and consequently SO-sequence. Since $(\mathcal{H}, \mathcal{K})$ satisfies P-property, for any $n \in \mathbb{N}$, we have

$$\begin{cases} \mathfrak{d}(\mathfrak{a}_{n+1}, \mathcal{S}\mathfrak{a}_n) = \mathfrak{d}(\mathcal{H}, \mathcal{K}), \\ \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) = \mathfrak{d}(\mathcal{H}, \mathcal{K}) \end{cases} \Longrightarrow \mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}). \tag{3.3}$$

Claim: $\{\mathfrak{a}_n\}$ is a Cauchy SO-sequence.
Now, since $\mathfrak{a}_0 \perp \mathfrak{a}_1$, we have $\mathfrak{d}(\mathfrak{a}_2, \mathfrak{a}_1) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_1, \mathcal{S}\mathfrak{a}_0) \leq \chi(\mathfrak{d}(\mathfrak{a}_1, \mathfrak{a}_0))$ and since $\mathfrak{a}_1 \perp \mathfrak{a}_2$, we have

$$\mathfrak{d}(\mathfrak{a}_3, \mathfrak{a}_2) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_2, \mathcal{S}\mathfrak{a}_1) \leq \chi(\mathfrak{d}(\mathfrak{a}_2, \mathfrak{a}_1)) \leq \chi^2(\mathfrak{d}(\mathfrak{a}_1, \mathfrak{a}_0)).$$

By induction

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi^n(\mathfrak{d}(\mathfrak{a}_1, \mathfrak{a}_0)) \to 0$$

as $n \to \infty$. Let $\zeta > 0$ be fixed. Choose $N \in \mathbb{N}$ so that

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) < \zeta - \chi(\zeta),\ \forall n \geq N. \tag{3.4}$$

We denote a ball with center x and radius ζ by $B[\mathfrak{a},\zeta]$. Since $\mathfrak{a}_{N+1} \in B[\mathfrak{a}_N,\zeta]$, so $\mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N) < \zeta$. Thus, from (3.1) and (3.3), we have

$$\begin{aligned} \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+1},\mathcal{S}\mathfrak{a}_{N-1}) &\leq \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+1},\mathcal{S}\mathfrak{a}_N) + \mathfrak{d}(\mathcal{S}\mathfrak{a}_N, T\mathfrak{a}_{N-1}) \\ &\leq \chi(\mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N)) + \mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N) < \chi(\zeta) + (\zeta - \chi(\zeta)) = \zeta. \end{aligned} \tag{3.5}$$

Therefore, $\mathcal{S}\mathfrak{a}_{N+1} \in B[\mathcal{S}\mathfrak{a}_{N-1},\zeta]$. From (3.2), $\mathfrak{d}(\mathfrak{a}_{N+2},\mathcal{S}\mathfrak{a}_{N+1}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$ with $\mathfrak{a}_{N+2} \in \mathcal{H}_0$ and $\mathfrak{d}(\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$. From (3.3), we have $\mathfrak{d}(\mathfrak{a}_{N+2},\mathfrak{a}_N) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+1},\mathcal{S}\mathfrak{a}_{N-1})$. By (3.5), $\mathfrak{d}(\mathfrak{a}_{N+2},\mathfrak{a}_N) < \zeta$, so $\mathfrak{a}_{N+2} \in B[\mathfrak{a}_N,\zeta]$ with $\mathfrak{a}_{N+2} \in \mathcal{H}_0$, therefore

$$\mathfrak{a}_{N+2} \in B[\mathfrak{a}_N,\zeta] \cap \mathcal{H}. \tag{3.6}$$

Again, from (3.1), (3.3) and (3.6), we get

$$\begin{aligned} \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+2},\mathcal{S}\mathfrak{a}_{N-1}) &\leq \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+2},\mathcal{S}\mathfrak{a}_N) + \mathfrak{d}(\mathcal{S}\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) \\ &\leq \chi(\mathfrak{d}(\mathfrak{a}_{N+2},\mathfrak{a}_N)) + \mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N) < \chi(\zeta) + (\zeta - \chi(\zeta)) = \zeta. \end{aligned} \tag{3.7}$$

Therefore, $\mathcal{S}\mathfrak{a}_{N+2} \in B[\mathcal{S}\mathfrak{a}_{N-1},\zeta]$. From (3.2), $\mathfrak{d}(\mathfrak{a}_{N+3},\mathcal{S}\mathfrak{a}_{N+2}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$ with $\mathfrak{a}_{N+3} \in \mathcal{H}_0$ and $\mathfrak{d}(\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$. From (3.3), we get $\mathfrak{d}(\mathfrak{a}_{N+3},\mathfrak{a}_N) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+2},\mathcal{S}\mathfrak{a}_{N-1})$. By (3.7), $\mathfrak{d}(\mathfrak{a}_{N+3},\mathfrak{a}_N) < \zeta$. So $\mathfrak{a}_{N+3} \in B[\mathfrak{a}_N,\zeta]$ with $\mathfrak{a}_{N+3} \in \mathcal{H}_0$, therefore $\mathfrak{a}_{N+3} \in B[\mathfrak{a}_N,\zeta] \cap A$. Now, again

$$\begin{aligned} \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+3},\mathcal{S}\mathfrak{a}_{N-1}) &\leq \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+3},\mathcal{S}\mathfrak{a}_N) + \mathfrak{d}(\mathcal{S}\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) \\ &\leq \chi(\mathfrak{d}(\mathfrak{a}_{N+3},\mathfrak{a}_N)) + \mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N) < \chi(\zeta) + (\zeta - \chi(\zeta)) = \zeta. \end{aligned} \tag{3.8}$$

Therefore, $\mathcal{S}\mathfrak{a}_{N+3} \in B[\mathcal{S}\mathfrak{a}_{N-1},\zeta]$. From (3.2), $\mathfrak{d}(\mathfrak{a}_{N+4},\mathcal{S}\mathfrak{a}_{N+3}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$ with $\mathfrak{a}_{N+4} \in \mathcal{H}_0$ and $\mathfrak{d}(\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) = \mathfrak{d}(\mathcal{H},\mathcal{K})$ and from (3.3), we get $\mathfrak{d}(\mathfrak{a}_{N+4},\mathfrak{a}_N) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+3},\mathcal{S}\mathfrak{a}_{N-1})$. By (3.8), $\mathfrak{d}(\mathfrak{a}_{N+4},\mathfrak{a}_N) < \zeta$. So $\mathfrak{a}_{N+4} \in B[\mathfrak{a}_N,\zeta]$ with $\mathfrak{a}_{N+4} \in \mathcal{H}_0$, therefore $\mathfrak{a}_{N+4} \in B[\mathfrak{a}_N,\zeta] \cap A$. Continuing this process, we have

$$\begin{aligned} \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+n+1},\mathcal{S}\mathfrak{a}_{N-1}) &\leq \mathfrak{d}(\mathcal{S}\mathfrak{a}_{N+n+1},\mathcal{S}\mathfrak{a}_N) + \mathfrak{d}(\mathcal{S}\mathfrak{a}_N,\mathcal{S}\mathfrak{a}_{N-1}) \\ &\leq \chi(\mathfrak{d}(\mathfrak{a}_{N+n+1},\mathfrak{a}_N)) + \mathfrak{d}(\mathfrak{a}_{N+1},\mathfrak{a}_N) < \chi(\zeta) + (\zeta - \chi(\zeta)) = \zeta. \end{aligned}$$

So, we can conclude that

$$\mathfrak{a}_{N+m} \in B[\mathfrak{a}_N,\zeta] \cap \mathcal{H},\ \forall m \in \mathbb{N}.$$

Then we get $\{\mathfrak{a}_n\}$ is a Cauchy SO-sequence in $\mathcal{H}$.

Since $\{\mathfrak{a}_n\}$ is a Cauchy SO-sequence in $\mathcal{H}$, $\mathcal{M}$ is a SO-complete and $\mathcal{H}$ is a closed subset of $\mathcal{M}$, the SO-sequence $\{\mathfrak{a}_n\}$ convergence to $\mathfrak{a}^* \in \mathcal{M}$. Since $\mathcal{S}$ is a SO-continuous map on $\mathcal{H}$, we have $\mathcal{S}\mathfrak{a}_n \to \mathcal{S}\mathfrak{a}^*$. By the continuity of the mapping $\mathfrak{d}$, we get $\mathfrak{d}(\mathfrak{a}_{n+1}, \mathcal{S}\mathfrak{a}_n) \to \mathfrak{d}(\mathfrak{a}^*, \mathcal{S}\mathfrak{a}^*)$. But (3.2) shows that sequence is a constant sequence converges to $\mathfrak{d}(\mathcal{H}, \mathcal{K})$. Therefore, $\mathfrak{d}(\mathfrak{a}^*, \mathcal{S}\mathfrak{a}^*) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$; that is, $\mathfrak{a}^* \in \mathcal{H}$ is a best proximity point for $\mathcal{S}$. □

Example. Let $\mathcal{M} = \mathbb{R}^2$. Define $\mathfrak{d} : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ by

$$d\big((\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2)\big) = (\mathfrak{a}_1 - \mathfrak{a}_2) + (\mathfrak{b}_1 - \mathfrak{b}_2)$$

for all $(\mathfrak{a}_1, \mathfrak{b}_1), (\mathfrak{a}_2, \mathfrak{b}_2) \in \mathbb{R}^2$. Let

$$\mathcal{H} = \big\{(\mathfrak{a}, 1) : \ \mathfrak{a} \in [0, 1]\big\} \quad \text{and} \quad \mathcal{K} = \big\{(\mathfrak{b}, 0) : \mathfrak{b} \in [0, 1]\big\}.$$

Clearly, $\mathfrak{d}(\mathcal{H}, \mathcal{K}) = 1$, $\mathcal{H} = \mathcal{H}_0$ and $\mathcal{K} = \mathcal{K}_0$. In particular $\mathcal{H}_0$ is nonempty.
Define binary relation $\perp$ on $\mathcal{M}$ by $(\mathfrak{a}_1, \mathfrak{b}_1) \perp (\mathfrak{a}_2, \mathfrak{b}_2)$ if $(|\mathfrak{a}_1 - \mathfrak{a}_2| \geq \frac{5}{6}, \mathfrak{b}_1 = \mathfrak{b}_2)$ and also define the mapping $\mathcal{S} : \mathcal{H} \to \mathcal{K}$ by

$$\mathcal{S}(\mathfrak{a}, 1) = \begin{cases} (0, 0) & 0 \leq \mathfrak{a} < 1, \\ (\frac{1}{3}, 0) & \mathfrak{a} = 1 \end{cases} \qquad \big(x \in [0, 1]\big).$$

Define the mapping $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\chi(\theta) = \begin{cases} \theta & 0 \leq \theta < 1, \\ 1 & \textit{otherwise.} \end{cases}$$

Observe that $\mathcal{S}$ is a $\perp$-preserving and χ non-decreasing function with $\lim_{n\to\infty} \chi^n(\theta) = 0$, for each $\theta > 0$. Let $(\mathfrak{a}_1, 1), (\mathfrak{a}_2, 1) \in \mathcal{H}$ and $(\mathfrak{a}_1, 1) \perp (\mathfrak{a}_2, 1)$, we have

- If $\mathfrak{a}_1, \mathfrak{a}_2 \in [0, 1)$ and $\mathfrak{a}_1 > \mathfrak{a}_2$, so

$$\begin{aligned} \mathfrak{d}\big(\mathcal{S}(\mathfrak{a}_1, 1), \mathcal{S}(\mathfrak{a}_2, 1)\big) &= \mathfrak{d}\big((0, 0), (0, 0)\big) = 0 \\ &\leq (\mathfrak{a}_1 - \mathfrak{a}_2) = \chi(\mathfrak{a}_1 - \mathfrak{a}_2) = \chi\big(\mathfrak{d}((\mathfrak{a}_1, 1), (\mathfrak{a}_2, 1))\big). \end{aligned}$$

- If $\mathfrak{a}_1 \vee \mathfrak{a}_2 = 1$ (let $\mathfrak{a}_2 = 1$), so $|\mathfrak{a}_1 - \mathfrak{a}_2| \geq \frac{5}{6}$ and

$$\begin{aligned} \mathfrak{d}\big(\mathcal{S}(\mathfrak{a}_1, 1), \mathcal{S}(1, 1)\big) &= \mathfrak{d}\big((0, 0), (\frac{1}{3}, 0)\big) = \frac{1}{3} \\ &\leq \frac{5}{6} \leq 1 = \chi(\mathfrak{a}_1 - 1) = \chi\big(\mathfrak{d}((\mathfrak{a}_1, 1), (1, 1))\big). \end{aligned}$$

According to Equation (3.1), $\mathcal{S}$ satisfies the necessary conditions. Furthermore, all the assumptions of Theorem 3.2 are satisfied, which implies that $\mathcal{S}$ has a unique best proximity point $\mathfrak{a} = (0, 1)$.

Theorem 3.3. *Assume that $(\mathcal{M}, \perp, \mathfrak{d})$ is strongly complete orthogonal Riemannian metric space and $(\mathcal{H}, \mathcal{K})$ be a pair of closed non-empty subset of the space $(\mathcal{M}, \perp, \mathfrak{d})$ with $\mathcal{H}_0 \neq \emptyset$. Let $\mathcal{S} : \mathcal{H} \longrightarrow \mathcal{K}$ be a map which*

i) $\mathcal{S}$ is a $\perp$-proximally increasing and $\perp$-preserving such that $\mathcal{S}(\mathcal{H}_0) \subseteq \mathcal{K}_0$, $(\mathcal{H}, \mathcal{K})$ satisfies the P-property;

ii) The orthogonal elements $\mathfrak{a}_0$ and $\mathfrak{a}_1$ exist in $\mathcal{H}_0$ such that $\mathfrak{d}(\mathfrak{a}_1, \mathcal{S}\mathfrak{a}_0) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$;

iii) $\mathcal{S}$ is a SO-continuous function on $\mathcal{H}$ such that

$$\mathfrak{d}(\mathcal{S}\mathfrak{a}, \mathcal{S}y) \leq \chi(\mathcal{Q}(\mathfrak{a}, \mathfrak{b})), \tag{3.9}$$

where

$$\mathcal{Q}(\mathfrak{a}, \mathfrak{b}) = \max\{\mathfrak{d}(\mathfrak{a}, \mathfrak{b}), \mathfrak{d}(\mathfrak{a}\mathcal{S}\mathfrak{a}) - \mathfrak{d}(\mathcal{H}, \mathcal{K}), \mathfrak{d}(\mathfrak{b}\mathcal{S}\mathfrak{b}) - \mathfrak{d}(\mathcal{H}, \mathcal{K}), \\ \frac{1}{2}[\mathfrak{d}(\mathfrak{a}, \mathcal{S}\mathfrak{b}) + \mathfrak{d}(y, \mathcal{S}\mathfrak{a}) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})], \frac{1}{2}[\mathfrak{d}(\mathfrak{a}, \mathcal{S}\mathfrak{a}) + \mathfrak{d}(\mathfrak{b}, \mathcal{S}\mathfrak{b}) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})]\},$$

for any point $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathcal{H}$ such that $\mathfrak{a} \perp \mathfrak{b}$ and $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ non-decreasing function with $\lim_{n\to\infty} \chi^n(t) = 0$, for each $t > 0$.

Then $\mathcal{S}$ has a best proximity point $\mathfrak{a}^* \in \mathcal{H}$ such that $\mathfrak{d}(\mathfrak{a}^*, \mathcal{S}\mathfrak{a}^*) = \mathfrak{d}(\mathcal{H}, \mathcal{K})$.

Proof. Similar to proof of Theorem 3.2, we shall prove that $\{\mathfrak{a}_n\}$ is a Cauchy SO-sequence in Riemannian metric space $(\mathcal{M}, \mathfrak{d})$.
Now, since $(\mathcal{H}, \mathcal{K})$ satisfies P-property and by (3.9), we have

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) = \mathfrak{d}(\mathcal{S}\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) \leq \chi(\mathcal{Q}(\mathfrak{a}_n, \mathfrak{a}_{n-1}))$$

where

$$\mathcal{Q}(\mathfrak{a}_n, \mathfrak{a}_{n-1}) = \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_n) - \mathfrak{d}(\mathcal{H}, \mathcal{K}), \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_{n-1}) - \mathfrak{d}(\mathcal{H}, \mathcal{K}), \\ \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_n) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})], \\ \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_{n-1}) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})]\}.$$

We now claim

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi(\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1})). \tag{3.10}$$

Now, by using (3.2) and (3.3), we have

$$\begin{aligned}
\mathcal{Q}(\mathfrak{a}_n, \mathfrak{a}_{n-1}) &= \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_n) - \mathfrak{d}(\mathcal{H}, \mathcal{K}), \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_{n-1}) - \mathfrak{d}(\mathcal{H}, \mathcal{K}),\\
&\quad \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_n) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})],\\
&\quad \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_{n-1}) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})]\}\\
&\leq \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}) + \mathfrak{d}(\mathfrak{a}_{n+1}, \mathcal{S}\mathfrak{a}_n) - \mathfrak{d}(\mathcal{H}, \mathcal{K}),\\
&\quad \mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) - \mathfrak{d}(\mathcal{H}, \mathcal{K}),\\
&\quad \frac{1}{2}[\mathfrak{d}(\mathcal{H}, \mathcal{K}) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) + \mathfrak{d}(\mathcal{S}\mathfrak{a}_{n-1}, \mathcal{S}\mathfrak{a}_n) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})],\\
&\quad \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}) + \mathfrak{d}(\mathfrak{a}_{n+1}, T\mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_n, \mathcal{S}\mathfrak{a}_{n-1}) - 2\mathfrak{d}(\mathcal{H}, \mathcal{K})]\}\\
&= \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}), \mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n)\\
&\quad \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})], \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}) + \mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n)]\}\\
&= \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}), \frac{1}{2}[\mathfrak{d}(\mathfrak{a}_{n-1}, \mathfrak{a}_n) + \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})]\}\\
&\leq \max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}
\end{aligned}$$

Thus, using the above inequality, (3.2), (3.3) and (3.9), we get

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi(\max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})\}), \quad \forall n \in \mathbb{N}.$$

Suppose that $\max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})\} = \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})$ and since $\chi(\theta) < \theta$ for each $\theta > 0$, we get

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi(\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})) < \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1}),$$

that is a contraction. So, we obtain

$$\max\{\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n+1})\} = \mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1}), \quad \forall n \in \mathbb{N}.$$

Thus

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi(\mathfrak{d}(\mathfrak{a}_n, \mathfrak{a}_{n-1})), \quad \forall n \in \mathbb{N}.$$

So, (3.10) holds. By induction, we have

$$\mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) \leq \chi^n(\mathfrak{d}(\mathfrak{a}_1, \mathfrak{a}_0)), \quad \forall n \in \mathbb{N}.$$

So $\lim_{n\to\infty} \mathfrak{d}(\mathfrak{a}_{n+1}, \mathfrak{a}_n) = 0$. To continue with the proof, we can follow a similar approach as in Theorem 3.2. □

4. Conclusion

Here, we defined several contractive mappings and showed the existence of their best proximity point. Several corollaries and comparisons are also explained in former sections to show the importance of main theorems, which extend many previous papers [11, 14].

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