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NEW SUBCLASSES OF U-SPACES

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Abstract. A topological space X is called a *U-space* if any two disjoint cozero-sets can be separated by a clopen set. The paper introduces and studies two new subclasses of U-spaces, which generalize the classes P-spaces and basically disconnected spaces. Some algebraic and topological characterizations of the new spaces are given. The relationship between these spaces is described and some counterexamples are presented which show that these spaces are different from each other.

1. Introduction

Several methods exist for constructing new topological spaces from old ones, e.g., products, sums, quotients, and projective limits. In this paper, during our investigations on U-spaces, we considered a new approach for constructing new spaces, and we hope the work we do here might enable interesting future research. Let us explain the basic idea of the work. We were led to the topic of this paper by thinking about the *zero-set* and *cozero-set* of a continuous function $f : X \to \mathbb{R}$, that is, $Z(f) = \{x \in X : f(x) = 0\}$ and $\operatorname{coz} f = X \setminus Z(f)$ respectively. It is known that the properties of some topological spaces can be expressed using zero-sets. For instance, a space X is said to be a *P-space*, if every zero-set in X is open or equivalently if every cozero-set in X is closed. For another

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example, a space X is said to be *basically disconnected* if every cozero-set in X has an open closure. Let us take a moment to write more explicitly about our idea in this study.

First suppose that $f \in C(X)$, where $C(X)$ will henceforth denote the ring of all real-valued continuous functions on a completely regular Hausdorff (i.e., Tychonoff) space X. As in [13], we denote the *positive part* of f by $\operatorname{pos} f = \{x \in X : f(x) > 0\}$ and the *negative part* of f by $\operatorname{neg} f = \{x \in X : f(x) < 0\}$. It is easy to see that $\operatorname{pos} f$ and $\operatorname{neg} f$ are cozero-sets, in fact $\operatorname{coz}(f^-) := \operatorname{coz}(f - |f|) = \operatorname{neg} f$, $\operatorname{coz}(f^+) := \operatorname{coz}(f + |f|) = \operatorname{pos} f$, and $\operatorname{coz} f = \operatorname{pos} f \cup \operatorname{neg} f$. The converse is also true, i.e., every cozero-set is a positive or negative part of an element of $C(X)$. To this end, for each $f \in C(X)$, we have $\operatorname{coz} f = \operatorname{pos} |f| = \operatorname{neg}(-|f|)$. The motivation for this paper comes from the following question: Is it possible to describe spaces characteristics by using at least one part of $\operatorname{coz} f$? For the main purposes of this paper, we are going to investigate the spaces X such that either $\operatorname{pos} f$ or $\operatorname{neg} f$ is closed (has an open closure) for each $f \in C(X)$. In comparison with P-spaces and basically disconnected spaces, let us call them *semi-P-space* and *semi-basically disconnected spaces*. More precisely, we call a space X a semi-P-space (resp., semi-basically disconnected space) if for each $f \in C(X)$ either $\operatorname{pos} f$ or $\operatorname{neg} f$ is closed (resp., has an open closure).

The paper is organized as follows: In Section 2, we present a thorough list, including some new equivalences for U-spaces, in hopes that there will be a better understanding of this interesting class of spaces. In Section 3, we give some algebraic and topological characterizations for a space X to be a semi-P-space and we observe that semi-P-spaces are precisely F-spaces in which all but at most one point is a P-point. In Section 4, we do the same for a semi-basically disconnected space. Finally, the short Section 5 devotes to a diagram concerning the relations between these spaces and we present counterexamples to show that these spaces are different.

Throughout this paper, all topological spaces are completely regular Hausdorff (i.e., Tychonoff) spaces, and rings are commutative with $1 \neq 0$. The annihilator of a subset A of a ring R is denoted by $\operatorname{Ann}(A) = \{r \in R | rA = 0\}$. Following the tradition, we use $\operatorname{Ann}(a)$ for $\operatorname{Ann}(\{a\})$. When $\operatorname{Ann}(a) \neq 0$, we say a is a *zerodivisor*; otherwise, we call it *regular* (i.e., *non-zerodivisor*). For each subset S of a ring R, let P_S (resp., M_S) be the intersection of all minimal prime (resp., maximal) ideals containing S. Every maximal ideal of $C(X)$ is precisely of the form $M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$, $p \in$

βX, where βX is the *Stone-Čech compactification* of X, see [13, Theorem 7.3]. The prime ideals containing a given prime ideal form a chain, so that every prime ideal is contained in a unique maximal ideal M^p, for a unique $p \in \beta X$; and the intersection of all the prime ideals contained in M^p is the ideal $O^p = \{f \in C(X) : p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\} = \{f \in C(X) : gf = 0, \text{for some}\, g \in C(X) \setminus M^p\}$. For a subset $A \subseteq \beta X$, M^A (resp., O^A) is the intersection of all M^p (resp., O^p), where p runs through A. In particular, if $A \subseteq X$, we denote M^A (resp., O^A) by M_A (resp., O_A). In case $A = Z(f)$ for some $f \in C(X)$, M_A is denoted by M_f and it is the intersection of all maximal ideals containing f. In fact, $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. Similarly, the intersection of all minimal prime ideals of $C(X)$ containing f is denoted by P_f and it is easy to see that $P_f = \{g \in C(X) : \mathrm{int}_X Z(f) \subseteq \mathrm{int}_X Z(g)\}$. We recall that υX is the Hewitt realcompactification of X and the reader is referred to [13] and [20] for undefined terms and notations.

2. A history of U-spaces

This section gives an account of the class of U-spaces. In Theorem 2.1, we give a list of characterizations of U-spaces. Most are old theorems but some new ones are also included. For completeness, we shall define most of the necessary concepts. To that end, we begin with the definition of a U-space.

A space X is called a *U-space* if for every $f \in C(X)$, there is a unit u of $C(X)$ such that $f = u|f|$ (whence $|f| = uf$). The definition of U-space is due to Gillman and Henriksen [14]. The class of U-spaces includes basically disconnected spaces (e.g., P-spaces and extremally disconnected spaces). In [14, Theorem 5.5], it is proved that a space X is a U-space if and only if it is a strongly zero-dimensional F-space. To give an exposition of [14, Theorem 5.5], we recall some definitions and facts below.

Recall that a space X is an *F-space* if every cozero-set in X is C^*-embedded, for more details, see [14, p. 208]. A ring R is called *Bézout* if every finitely generated ideal is principal. It is well-known that $C(X)$ is a Bézout ring if and only if X is an F-space, see [14, Theorem 14.25]. Conditions equivalent to $C(X)$ being a Bézout ring are given in [4, §5]. A space X is *strongly zero-dimensional* if for every pair of disjoint zero sets, Z_1 and Z_2, there is a clopen set B such that $Z_1 \subseteq B$ and $Z_2 \cap B = \emptyset$. It is known that a space X is strongly zero-dimensional if and only if βX is zero-dimensional. A ring R is *exchange* if for any $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$, see

[16] for more details. A ring R is said to be *clean* if each $x \in R$ can be written as $x = u + e$ where u is a unit and e is an idempotent. Nicholson [24] proved that an abelian ring (a ring in which all idempotents are central) is clean if and only if it is exchange. A ring R is called a *Gelfand ring* (also known as *pm-ring* [10]) if every prime ideal of R is contained in a unique maximal ideal. The rings $C(X)$ constitute an important class of Gelfand rings [13, Theorem 2.11]. In [17, p. 201], Johnstone proved that a commutative ring R is an exchange ring if and only if R is a Gelfand ring and $\mathrm{Max}(R)$ is zero-dimensional. Consequently, $C(X)$ is a clean ring if and only if X is strongly zero-dimensional. This characterization was rediscovered two decades later by some authors with different proofs, see [3, 6, 17, 22, 26].

Before presenting the main result of this section, we recall the definition of an almost p.p. ring [2]. A ring R is called an *almost p.p. ring* if for each $a \in R$; the annihilator ideal $\mathrm{Ann}(a)$ is generated by its idempotents (the terms *almost weak Baer* [25] and *feebly Baer* [18] are also used).

We are now ready to state the promised theorem.

Theorem 2.1. *The following statements are equivalent:*

(1) *X is a U-space.*
(2) *X is a strongly zero-dimensional F-space.*
(3) *$C(X)$ is a clean Bézout ring.*
(4) *$C(X)$ is an almost p.p-ring.*
(5) *If U and V are cozero-sets with $U \cap V = \emptyset$, then U and V can be separated by a clopen set.*
(6) *For every $f \in C(X)$ and $g \in \mathrm{Ann}(f)$, there exists an idempotent $e \in \mathrm{Ann}(f)$ such that $g = eg$.*
(7) *For every $f \in C(X)$, there is an idempotent e of $C(X)$ such that $f = (2e-1)|f|$.*
(8) *For every $f \in C(X)$, there is an idempotent e of $C(X)$ such that $f = -(2e-1)|f|$.*
(9) *For every $f \in C(X)$, $\mathrm{pos}\, f$ and $\mathrm{neg}\, f$ can be separated by a clopen set.*

Proof. (1) ⇔ (2) It follows from [14, Theorem 5.5] (or [18, Theorem 5.7]).
(2) ⇔ (3) This is clear by the comments before the theorem.
(1) ⇔ (4) It follows from [2, Theorem 2.4].
(1) ⇔ (5) It follows from [7, Proposition 4.5].

(4) $\Leftrightarrow$ (6) It follows from [25, Lemma 1.2 and Proposition 1.3] (or [18, Theorem 2.13]).
(6) $\Rightarrow$ (7) Let $f \in C(X)$. From (6), there exists $e^2 = e$ such that $(f - |f|)e = 0$ and $f + |f| = (f + |f|)e$. Hence, we have $fe = |f|e$ and $f + |f| = fe + |f|e$. Thus, we infer that $f + |f| = |f|e + |f|e = 2|f|e$. This yields that $f = (2e - 1)|f|$, as desired.
(7) $\Rightarrow$ (1) It follows from the fact that $(2e - 1)^2 = 1$ for each idempotent e.
(7) $\Leftrightarrow$ (8) It suffices to take $e' = 1 - e$.
(7) $\Leftrightarrow$ (9) It follows from the fact that $f = (2e - 1)|f|$ if and only if $\operatorname{neg} f \subseteq Z(e)$ and $\operatorname{pos} f \subseteq X \setminus Z(e)$, for any f and $e^2 = e$. □

3. **Semi-P-spaces**

In the context of $C(X)$, a P-space is one of the topology's most important notions ever introduced. A point $x \in X$ is called a *P-point* if $x \in Z(f)$, $f \in C(X)$ implies $x \in \operatorname{int}_X Z(f)$ and X is said to be a *P-space* if every point of X is a P-point. Thus a space X is a P-space if every zero-set of X is open or every cozero-set in X is closed. Conditions equivalent to a space being a P-space are given in [13, 4J and 14.29] and [4, §3]. It is known that X is a P-space if and only if $C(X)$ is a von Neumann regular ring, see [13, 4J(8)]. Recall that an element a in a ring R is called *von Neumann regular* if $a = a^2b$ for some $b \in R$, or equivalently, if there are a unit $u \in R$ and an idempotent $e \in R$ such that $a = ue$, see [11, Corollary 1] for example. A ring R is called von Neumann regular if every element is von Neumann regular.

As we mentioned in the introduction, we say a space X is a semi-P-space if for each $f \in C(X)$, either $\operatorname{pos} f$ or $\operatorname{neg} f$ is closed. Since every cozero-set is the positive (negative) part of an element of $C(X)$, obviously, every P-space is a semi-P-space. However, the converse is not true, see Example 5.1(3). Our aim here is to provide some characterizations of semi-P-spaces and using them we observe that semi-P-spaces coincide with P_F-spaces introduced in [5]. To achieve this goal we first prove a preliminary lemma. Next, we introduce the concept of "semi-z-ideals" as a generalization of z-ideals.

Lemma 3.1. *The following statements hold.*

(1) *X is a P-space if and only if for each $f \in C(X)$, $Z(f + |f|)$ and $Z(f - |f|)$ are open.*

(2) *For* $f \in C(X)$, $Z(f)$ *is open if and only if there are a unit* u *and an idempotent* e *such that* $f = ue$.

(3) *For each* $f \in C(X)$, $Z(f + |f|)$ *is open if and only if there is an idempotent* e *of* $C(X)$ *such that* $f = (2e - 1)|f|$ *and* $Z(fe)$ *is open.*

(4) *For each* $f \in C(X)$, $Z(f - |f|)$ *is open if and only if there is an idempotent* e *of* $C(X)$ *such that* $f = -(2e - 1)|f|$ *and* $Z(fe)$ *is open*

Proof. (1) The left-to-right implication immediate and the converse follows from the fact that $Z(f) = Z(f + |f|) \cap Z(f - |f|)$.

(2) Assume that $Z(f)$ is open. Define

$$u(x) = \begin{cases} 1 & x \in Z(f) \\ 1/f & x \notin Z(f). \end{cases}$$

Obviously, u is a unit and $f^2u = f$. Let $e = fu$. It is clear that e is an idempotent and $eu^{-1} = f$, as desired. Conversely, if $f = ue$, we deduce that $Z(f) = Z(e)$ and so $Z(f)$ is open.

(3) Assume that $Z(f + |f|)$ is open. There are a unit u and an idempotent e such that $f + |f| = ue$ by part (2). Since $f - |f| \in \mathrm{Ann}(f + |f|)$, we have $(f - |f|)ue = 0$, whence $(f - |f|)e = 0$. Hence, $fe - |f|e = 0$ which implies that $f + |f| = 2e|f|$. From this, we conclude that $f = (2e - 1)|f|$ and $Z(f + |f|) = Z(ef)$ is open. For the converse, we have $f + |f| = 2e|f|$. From $Z(f + |f|) = Z(fe)$, we infer that $Z(f + |f|)$ is open.

(4) The proof is similar to (3). □

Recall that an ideal I of $C(X)$ is a *z-ideal* if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ imply that $g \in I$. The notion of z-ideals was first introduced by Kohls [19]. More information about z-ideals can be found in [13] and [21].

Definition 3.2. We say an ideal I of $C(X)$ is a *semi-z-ideal* if for every pair of elements $f, g \in C(X)$, one of the following conditions hold:

(1) $Z(f + |f|) \subseteq Z(g + |g|)$ and $f + |f| \in I$ imply that $g + |g| \in I$.

(2) $Z(f - |f|) \subseteq Z(g - |g|)$ and $f - |f| \in I$ imply that $g - |g| \in I$.

These are equivalent to saying respectively that:

(1′) $\mathrm{pos}\, g \subseteq \mathrm{pos}\, f$ and $f + |f| \in I$ imply that $g + |g| \in I$.

(2′) $\mathrm{neg}\, g \subseteq \mathrm{neg}\, f$ and $f - |f| \in I$ imply that $g - |g| \in I$.

Obviously, every z-ideal is a semi-z-ideal. However, the following example shows that the converse is not true in general. Before giving the example, let us recall that an ideal I of $C(X)$ is *pseudoprime* if for $f, g \in C(X)$, $fg = 0$ implies $f \in I$ or $g \in I$, see [15, 3.1]. Clearly, every prime ideal of $C(X)$ is pseudoprime. However, the converse is not true, see [15, 4.5 and 4.6].

Remark 3.3. We may also define semi-z-ideal in $C(X)$ as follows: An ideal I of $C(X)$ is a semi-z-ideal if for every $f, g \in C(X)$ with $f + |f|, f - |f| \in I$, whenever $Z(f + |f|) \subseteq Z(g + |g|)$ and $Z(f - |f|) \subseteq Z(g - |g|)$, then either $g + |g| \in I$ or $g - |g| \in I$. Using this form of the definition, it is easy to see that every pseudoprime ideal of $C(X)$ is a semi-z-ideal.

Example 3.4. Take a prime ideal P that is not a z-ideal, see for example [13, 2G.1]. It is clear that P is a semi-z-ideal that is not a z-ideal.

The following example establishes the existence of an ideal of $C(X)$ that is not a semi-z-ideal.

Example 3.5. Let i be the identity function in $C(\mathbb{R})$. We claim that the principal ideal $I =< i >$ is not a semi-z-ideal. Assume, for a contradiction, I is a semi-z-ideal. We note that $\operatorname{pos} i = \operatorname{pos} i^3$ and $\operatorname{neg} i = \operatorname{neg} i^3$ and $i^3 \pm |i^3| \in I$. By [13, 2H], $i + |i| \notin I$ and $i - |i| \notin I$ because $i \pm |i|$ are not differentiable at 0. That is a contradiction.

We are now able to prove the following. First from [5], whenever, of any two zero-sets of a space X whose union is all of X at least one is open, then we call the space X a P_F-space. It is shown in the same reference that any P_F-space is precisely an F-space in which all but at most one point is a P-point, see Theorem 2.4 in [5]. Using Proposition 2.5 in [5] and the following theorem, every pseudocompact (compact) semi-P-space is finite.

Theorem 3.6. *The following statements are equivalent:*

(1) *X is a semi-P-space.*
(2) *For each $f \in C(X)$, either $Z(f - |f|)$ or $Z(f + |f|)$ is open.*
(3) *X is an F-space in which at most one point fails to be a P-point.*
(4) *For each $f \in C(X)$, either $\operatorname{pos} f$ or $\operatorname{neg} f$ is C-embedded.*
(5) *For each $f \in C(X)$, either $f - |f| \in O_{Z(f)}$ or $f + |f| \in O_{Z(f)}$.*

(6) *For each* $f \in C(X)$*, there is a unit* $u \in C(X)$ *such that* u *is constant on* $Z(f)$ *and* $f = u|f|$.

(7) *For each* $f \in C(X)$*, there is an idempotent* $e \in C(X)$ *such that* e *or* $1-e$ *belongs to* M_f *and* $f = (2e-1)|f|$.

(8) *Every ideal of* $C(X)$ *is a semi-z-ideal.*

(9) *For each* $f \in C(X)$*, there is an idempotent* e *of* $C(X)$ *such that* $f = (2e-1)|f|$ *and either* $Z(fe)$ *or* $Z(f(1-e))$ *is open.*

(10) υX *is semi-P-space.*

Proof. (1) ⇔ (2) It follows from the fact that $\text{coz}(f-|f|) = \text{neg}\, f$ and $\text{coz}(f+|f|) = \text{pos}\, f$, for each $f \in C(X)$.

(1) ⇒ (3) ⇒ (2) Assume (1) and $Z(f) \cup Z(g) = X$, $f, g \in C(X)$. By the argument preceding the theorem, we must show that either $Z(f)$ or $Z(g)$ is open. Take $h = f^2 - g^2$. Then either $\text{pos}\, h$ or $\text{neg}\, h$ is closed by (1). But $\text{pos}\, h = \text{coz}\, f$ and $\text{neg}\, h = \text{coz}\, g$ since $fg = 0$. Therefore, either $\text{coz}\, f$ or $\text{coz}\, g$ is closed and we are done. Now suppose that (3) holds. Since $Z(f-|f|) \cup Z(f+|f|) = X$, either $Z(f-|f|)$ or $Z(f+|f|)$ is open by the argument preceding of the theorem, i.e., (2) holds.

(1) ⇒ (4) It is clear because either $\text{pos}\, f$ or $\text{neg}\, f$ is clopen.

(4) ⇒ (1) Let $f \in C(X)$. Assume that $\text{pos}\, f$ is C-embedded. Thus, $g = \frac{1}{f} \in C(\text{pos}\, f)$ has an extension in $C(X)$ which implies that $\text{cl}_X \text{pos}\, f = \text{pos}\, f$, i.e., $\text{pos}\, f$ is closed. A similar argument works when $\text{neg}\, f$ is C-embedded.

(2) ⇒ (5) Assume that $Z(f-|f|)$ is open. Hence, we have $Z(f) \subseteq Z(f-|f|) = \text{int}_X Z(f-|f|)$. This implies that $f-|f| \in O_{Z(f)}$. Similarly, if $Z(f+|f|)$ is open, we get $f+|f| \in O_{Z(f)}$.

(5) ⇒ (6) Assume that $f-|f| \in O_{Z(f)}$. From $Z(f) \subseteq \text{int}_X Z(f-|f|)$ and $\text{pos}\, f \subseteq \text{int}_X Z(f-|f|)$, we have $Z(f-|f|) = Z(f) \cup \text{pos}\, f \subseteq \text{int}_X Z(f-|f|)$. Hence, $Z(f-|f|)$ is open and we may consider $Z(f-|f|) = Z(e)$ for some idempotent $e \in C(X)$. Take $u = 1-2e$. It is easy to check that u is a unit, $u = 1$ on $Z(f)$ and $f = u|f|$. A similar argument works when $f+|f| \in O_{Z(f)}$.

(6) ⇒ (7) It is enough to take $e = \frac{u+1}{2}$. Clearly, e or $1-e$ belongs to M_f and $f = (2e-1)|f|$ by (6).

(7) $\Rightarrow$ (1) Assume $1-e \in M_f$. Using $f=(2e-1)|f|$, we have $e=1$ on $Z(f-|f|)$ and $e=0$ on $\operatorname{neg} f$. This implies that $Z(f-|f|)$ is open. If $e \in M_f$, then we observe similarly that $e=0$ on $Z(f+|f|)$ and $e=1$ on $\operatorname{pos} f$ which means that $Z(f+|f|)$ is open.

(1) $\Rightarrow$ (8) Without loss of generality, we may assume that $Z(f+|f|)$ is open. We show that the condition $(1')$ of Definition 3.2 holds. Suppose that $\operatorname{pos} g \subseteq \operatorname{pos} f$ and $f+|f| \in I$. Hence $Z(f+|f|) = X \setminus \operatorname{pos} f \subseteq X \setminus \operatorname{pos} g = Z(g+|g|)$. Since $Z(f+|f|)$ is open, we have $Z(f+|f|) \subseteq \operatorname{int}_X Z(g+|g|)$. Now using [13, 1D], $g+|g|$ is a multiple of $f+|f|$, whence $g+|g| \in I$, as desired. Whenever $Z(f-|f|)$ is open, we may similarly show that the condition $(2')$ of Definition 3.2 holds.

(8) $\Rightarrow$ (1) Let $f \in C(X)$ and the condition $(1')$ of Definition 3.2 holds. We claim that $Z(f+|f|)$ is open. Take $g=(f+|f|)^{\frac{1}{3}}$. Then $\operatorname{pos} g = \operatorname{pos} f$ and $f+|f| \in I =< f+|f| >$ imply that $g+|g| \in < f+|f| >$. Therefore, $(f+|f|)^{\frac{1}{3}} + |(f+|f|)^{\frac{1}{3}}| = 2(f+|f|)^{\frac{1}{3}} \in < f+|f| >$ which implies that $Z(f+|f|)$ is open, as desired. Whenever the condition $(2')$ of Definition 3.2 holds, we take $g=(f-|f|)^{\frac{1}{3}}$ and similarly show that $Z(f-|f|)$ is open.

(1) $\Leftrightarrow$ (9) It follows from Lemma 3.1.

(1) $\Leftrightarrow$ (10) Using Remark 8.8 in [13], $C(\upsilon X) \cong C(X)$. On the other hand for each $f \in C(X)$, clearly $(f \pm |f|)^{\upsilon} = f^{\upsilon} \pm |f^{\upsilon}|$, where f^{υ} is the extension of f in $C(\upsilon X)$. Now using these facts, the proof is evident. □

Remark 3.7. sing part (6) of the Theorem 3.6, every semi-P-space is a U-space and the equivalence of parts (1) and (3) of the theorem means that semi-P-spaces and P_F-spaces coincide.

We conclude this section by the following example of a semi-z-ideal in $C(X)$ that is neither a pseudoprime ideal nor a z-ideal.

Example 3.8. We consider the space $\Sigma = \mathbb{N} \cup \{\sigma\}$ in [13, 4M]. First it is clear by [13, 4M] that Σ is a semi-P-space. Next, let E and O be the sets of all even and odd integers respectively. Since the filter $\mathcal{U}$ in 4M is an ultrafilter, either E or O belongs to $\mathcal{U}$, say $E \in \mathcal{U}$. By [13, 4M. 1], there exists $f \in C(\Sigma)$ such that $Z(f) = O \cup \{\sigma\}$. Put $I =< f >$. Since Σ is a semi-P-space, the ideal I is a semi-z-ideal in $C(\Sigma)$ by Theorem 3.6. But I is not a z-ideal, it is not even semiprime because $Z(f)$ is not open ($O \notin \mathcal{U}$). Also I is not a pseudoprime ideal. In fact, if I is pseudoprime, then it contains a prime ideal P by Theorem 4.1 in [15]. Now $I \subseteq M_\sigma$ implies that $P \subseteq M_\sigma$ and hence $O_\sigma \subseteq P$, see Theorem

7.15 in [13]. This implies that $O_\sigma \subseteq I$ while $g \in C(\Sigma)$ with $Z(g) = E \cup \{\sigma\}$ is contained in O_σ but $g \notin I$, a contradiction.

4. **Semi-basically disconnected spaces**

Other interesting Tychonoff spaces are basically disconnected spaces. A space X is *basically disconnected* if the closure of every cozero-set is clopen, or equivalently, if the interior of every zero-set is closed. It is known that X is a basically disconnected space if and only if $C(X)$ is a p.p. ring, see [8, 9]. Recall that a ring R is said to be a *p.p. ring* (also known as *Rickart ring* [20]) if every principal ideal of R is projective, or equivalently, if the annihilator of each element of R is generated by an idempotent. Conditions equivalent to $C(X)$ being a p.p. ring are given in [4, §4]. As we defined in the introduction, we say a space X semi-basically disconnected if for each $f \in C(X)$, either $\mathrm{cl}_X \operatorname{pos} f$ or $\mathrm{cl}_X \operatorname{neg} f$ is open. Hence every semi-P-space is a semi-basically disconnected space but every semi-basically disconnected space, even every basically disconnected space need not be a semi-P-space, see Example 5.1(2). The following lemma shows that every basically disconnected space is semi-basically disconnected space. In Example 5.1(4) of Section 5 we will see that the converse is not true.

Lemma 4.1. *The following statements are equivalent:*

(1) *X is a basically disconnected space.*
(2) *For each $f \in C(X)$, $\mathrm{cl}_X \operatorname{pos} f$ and $\mathrm{cl}_X \operatorname{neg} f$ are open.*
(3) *For each $f \in C(X)$, $\mathrm{int}_X Z(f + |f|)$ and $\mathrm{int}_X Z(f - |f|)$ are closed.*

Proof. (1) ⇔ (2) It is clear because the positive and negative parts of every $f \in C(X)$ is a cozero-set and $\mathrm{cl}_X \operatorname{coz} f = \mathrm{cl}_X \operatorname{pos} f \cup \mathrm{cl}_X \operatorname{neg} f$.

(1) ⇔ (3) Clearly (1) implies (3) and the converse follows from the fact that $\mathrm{int}_X Z(f) = \mathrm{int}_X Z(f + |f|) \cap \mathrm{int}_X Z(f - |f|)$. □

We are going to give some characterizations of semi-basically disconnected spaces. First, we state the following lemma which should be known to the experts but cannot be found explicitly stated in the literature. This lemma is the counterpart of Lemma 3.1, we prove part (1) and leave the proof of parts (2) and (3) to the reader.

Lemma 4.2. *Let $f \in C(X)$. Then the following statements hold.*

(1) $\mathrm{int}_X\, Z(f)$ *is closed if and only if there are a regular element* r *and an idempotent* e *such that* $f = re$.
(2) $\mathrm{int}_X\, Z(f + |f|)$ *is closed if and only if there is an idempotent* e *of* $C(X)$ *such that* $f = (2e - 1)|f|$ *and* $\mathrm{int}_X\, Z(fe)$ *is closed.*
(3) $\mathrm{int}_X\, Z(f - |f|)$ *is closed if and only if there is an idempotent* e *of* $C(X)$ *such that* $f = -(2e - 1)|f|$ *and* $\mathrm{int}_X\, Z(fe)$ *is closed.*

Proof. (1) First we let $\mathrm{int}_X Z(f)$ be closed. Define

$$e(x) = \begin{cases} 0 & x \in \mathrm{int}_X\, Z(f) \\ 1 & x \notin \mathrm{int}_X\, Z(f). \end{cases}$$

$$r(x) = \begin{cases} 1 & x \in \mathrm{int}_X\, Z(f) \\ f(x) & x \notin \mathrm{int}_X\, Z(f). \end{cases}$$

Clearly $e^2 = e$, $\mathrm{int}_X Z(r) = \emptyset$ and $f = re$ (note, $Z(r) \subseteq Z(f) \setminus \mathrm{int}_X Z(f)$).

Next, assume that $f = re$, where r is a regular element and e is an idempotent. Then, we have $\mathrm{int}_X Z(f) = \mathrm{int}_X(Z(r) \cup Z(e)) = \mathrm{int}_X Z(e) = Z(e)$, i.e., $\mathrm{int}_X Z(f)$ is closed. □

Following [27], an element a in a ring R is called a *p.p. element* if a is a product of a regular and an idempotent in R. Using this definition we may restate parts (2) and (3) of Lemma 4.2 as follows: $\mathrm{int}_X\, Z(f + |f|)$ (resp., $\mathrm{int}_X\, Z(f - |f|)$) is closed if and only if $f + |f|$ (resp., $f - |f|$) is a p.p. element. A ring R is a p.p. ring, if every element of R is a p.p. element.

In view of Lemma 4.2, let us make the following observation for future reference.

Corollary 4.3. *For* $f \in C(X)$, $\mathrm{int}_X\, Z(f)$ *is closed if and only if* f *is a p.p. element.*

We can now prove the following.

Theorem 4.4. *The following statements are equivalent:*

(1) X *is a semi-basically disconnected space.*
(2) *For each* $f \in C(X)$, *either* $\mathrm{int}_X\, Z(f - |f|)$ *or* $\mathrm{int}_X\, Z(f + |f|)$ *is closed.*
(3) *For every pair of elements* $f, g \in C(X)$, $fg = 0$ *implies either* f *or* g *is a p.p. element.*
(4) *For each* $f \in C(X)$, *there is an idempotent* e *in* $C(X)$ *such that either* e *or* $1 - e$ *belongs to* P_f *and* $f = (2e - 1)|f|$.

(5) *For each* $f \in C(X)$*, there is a unit* u *in* $C(X)$ *such that* $f = u|f|$ *and* u *is constant on* $\operatorname{int}_X Z(f)$.
(6) *For each* $f \in C(X)$*, there is an idempotent* e *of* $C(X)$ *such that* $f = (2e-1)|f|$ *and either* fe *or* $f(1-e)$ *is a p.p. element.*
(7) *For each* $f \in C(X)$*, there is an idempotent* e *such that* $\operatorname{int}_X Z(fe)$ *is closed and* $\operatorname{pos} f$ *and* $\operatorname{neg} f$ *can be separated by* $Z(e)$.
(8) *For each* $f \in C(X)$ *there is a unit* u *of* $C(X)$ *such that* $f = u|f|$ *and either* $(1+u)f$ *or* $(1-u)f$ *is a p.p. element.*
(9) *For every pair of elements* $f, g \in C^*(X)$*,* $fg = 0$ *implies either* f *or* g *is a p.p. element.*
(10) βX *is a semi-basically disconnected space.*

Proof. (1) ⇔ (2) It follows from the fact that $X \setminus \operatorname{int}_X Z(f - |f|) = \operatorname{cl}_X \operatorname{coz}(f - |f|) = \operatorname{cl}_X \operatorname{neg} f$ and $X \setminus \operatorname{int}_X Z(f + |f|) = \operatorname{cl}_X \operatorname{coz}(f + |f|) = \operatorname{cl}_X \operatorname{pos} f$, for each $f \in C(X)$.

(1) ⇒ (3) Let $f, g \in C(X)$ and $fg = 0$. Take $h = f^2 - g^2$. Clearly, $\operatorname{pos} h = \operatorname{coz} f$ and $\operatorname{neg} h = \operatorname{coz} g$. From (1), we deduce that either $\operatorname{cl}_X \operatorname{pos} h$ or $\operatorname{cl}_X \operatorname{neg} h$ is open. Hence, either $\operatorname{cl}_X \operatorname{coz} f$ or $\operatorname{cl}_X \operatorname{coz} g$ is open, or equivalently, either f or g is a p.p. element by Corollary 4.3.

(3) ⇒ (2) Since $(f+|f|)(f-|f|) = 0$, either $f+|f|$ or $f-|f|$ is a p.p. element. Hence, either $\operatorname{int}_X Z(f+|f|)$ or $\operatorname{int}_X Z(f-|f|)$ is closed by Lemma 4.2.

(1) ⇒ (4) Let $f \in C(X)$. Without loss of generality, we may assume that $\operatorname{cl}_X \operatorname{pos} f$ is open. Thus, we have $\operatorname{cl}_X \operatorname{pos} f = Z(1-e)$ for some idempotent e of $C(X)$. Since $\operatorname{cl}_X \operatorname{pos} f \cup \operatorname{int}_X Z(f) \cup \operatorname{cl}_X \operatorname{neg} f = X$ and they are mutually disjoint, we have $\operatorname{int}_X Z(f) \cup \operatorname{cl}_X \operatorname{neg} f = Z(e)$. Now it is clear that $f = (2e-1)|f|$ and $e \in P_f$. A similar argument works when $\operatorname{cl}_X \operatorname{neg} f$ is open.

(4) ⇒ (5) Take $u = 2e - 1$. Then $u^2 = 1$, $u - 1 = 2(e-1)$ and $u + 1 = 2e$. Thus, (4) implies that $f = u|f|$ and either $u+1$ or $u-1$ belongs to P_f which means that $u = 1$ on $\operatorname{int}_X Z(f)$ or $u = -1$ on $\operatorname{int}_X Z(f)$.

(5) ⇒ (1) Assume (5). From $u^2 = 1$, we infer that $Z(u+1)$ and $Z(u-1)$ are open sets. If $u = 1$ on $\operatorname{int}_X Z(f)$ and $f = u|f|$, then $\operatorname{int}_X Z(f) \subseteq Z(u-1)$ and $\operatorname{cl}_X \operatorname{pos} f \subseteq Z(u-1)$. Hence, $Z(u+1) \subseteq X \setminus (\operatorname{int}_X Z(f) \cup \operatorname{cl}_X \operatorname{pos} f) = \operatorname{cl}_X \operatorname{neg} f \subseteq Z(u+1)$, so $\operatorname{cl}_X \operatorname{neg} f$ is open. Whenever $u = -1$ on $\operatorname{int}_X Z(f)$, we observe similarly that $\operatorname{cl}_X \operatorname{pos} f$ is open.

(6) ⇔ (1) It follows from Corollary 4.3 and Lemma 4.2.

(7) $\Rightarrow$ (1) Assume (7). Without loss of generality, we may assume that $\operatorname{pos} f \subseteq Z(e)$. Then, it is clear that $\operatorname{pos} ef = \emptyset$, $\operatorname{neg} ef = \operatorname{neg} f$, and hence

$$X = \operatorname{int}_X Z(ef) \cup \operatorname{cl}_X \operatorname{pos} ef \cup \operatorname{cl}_X \operatorname{neg} ef = \operatorname{int}_X Z(fe) \cup \operatorname{cl}_X \operatorname{neg} f. \qquad (*)$$

Since $\operatorname{int}_X Z(fe)$ is closed and disjoint from $\operatorname{cl}_X \operatorname{neg} f$, we infer that $\operatorname{cl}_X \operatorname{neg} f$ is open.

(1) $\Rightarrow$ (7) Assume (1). Without loss of generality, we may assume that $\operatorname{cl}_X \operatorname{pos} f$ is open. Take $\operatorname{cl}_X \operatorname{pos} f = Z(1-e)$ for some idempotent e of $C(X)$. It is enough to show that $\operatorname{int}_X Z(fe)$ is closed. Clearly, $\operatorname{cl}_X \operatorname{pos} ef = \operatorname{cl}_X \operatorname{pos} f$ and $\operatorname{neg} ef = \emptyset$ for $\operatorname{neg} f \subseteq Z(e)$. Now according to $(*)$, $X = \operatorname{int}_X Z(fe) \cup \operatorname{cl}_X \operatorname{pos} f = \operatorname{int}_X Z(fe) \cup Z(1-e)$. Therefore, $\operatorname{int}_X Z(ef) = Z(e)$ is closed as desired.

(8) $\Leftrightarrow$ (6) It is straightforward.

(3) $\Rightarrow$ (9) Assume (3). Let $f, g \in C^*(X)$ with $fg = 0$. Then, either f or g is a p.p. element of $C(X)$, say f. Then, $f = re$ where r is a regular element of $C(X)$ and e is an idempotent. If $|f| \leq M$, we take $r^* = -M \wedge r \vee M$. Hence, r^* is a regular element of $C^*(X)$ and $f = r^*e$, i.e., f is a p.p. element of $C^*(X)$.

(9) $\Rightarrow$ (3) Assume (9). Let $f, g \in C(X)$ and $fg = 0$. Then, $\frac{f}{1+|f|}\frac{g}{1+|g|} = 0$ implies either $\frac{f}{1+|f|}$ or $\frac{g}{1+|g|}$ is a p.p. element of $C^*(X)$. Since $1+|f|$ and $1+|g|$ are units, we deduce that either f or g is a p.p. element of $C(X)$.

(9) $\Leftrightarrow$ (10) It follows from the fact that $C^*(X) \cong C(\beta X)$. □

Remark 4.5. Part (5) of Theorem 4.4 shows that every semi-basically disconnected space is a U-space. The converse is not true by Example 5.1(1).

We conclude this section by the following result. First, recall a space X is an *almost P-space* if every nonempty zero-set in X has a nonempty interioir. It is clear that a space X is an almost P-space if and only if each zero-set (cozero-set) in X is a regular closed (regular open). A closed (open) subset A of X is said to be regular closed (regular open) if $\operatorname{cl}_X\operatorname{int}_X A = A$ ($\operatorname{int}_X\operatorname{cl}_X A = A$). Since each cozero-set is $\operatorname{pos} f$ and also is $\operatorname{neg} g$ for some $f, g \in C(X)$ and conversely for each $f \in C(X)$, $\operatorname{pos} f$ and $\operatorname{neg} f$ are cozero-sets, it is equivalent to saying that a space X is an almost P-space if and only if $\operatorname{pos} f$ and $\operatorname{neg} f$ are regular open for each $f \in C(X)$.

Proposition 4.6. *Every semi-basically disconnected almost P-space is a semi-P-space.*

Proof. Let $f \in C(X)$. If $Z(f) = \emptyset$, then clearly both $\operatorname{pos} f$ and $\operatorname{neg} f$ are closed. Now suppose that $Z(f) \neq \emptyset$. Whereas X is an almost P-space, then $\operatorname{pos} f$ and $\operatorname{neg} f$ are regular open by the argument preceding proposition. On the other hand, since X is semi-basically disconnected, without loss of generality we assume that $\operatorname{cl}_X \operatorname{pos} f$ is open. This implies that $\operatorname{pos} f$ is closed, because $\operatorname{pos} f = \operatorname{int}_X \operatorname{cl}_X \operatorname{pos} f = \operatorname{cl}_X \operatorname{pos} f$. Thus X is a semi-P-space.

□

5. **Counterexamples**

A routine examination of the above definitions leads to the implications illustrated in the following diagram.

$$\begin{array}{ccccc}
P\text{-space} & \longrightarrow & \text{basically disconnected space} & & \\
\downarrow & & \downarrow & & \\
\text{semi-}P\text{-space} & \longrightarrow & \text{semi-basically disconnected space} & \longrightarrow & U\text{-space}
\end{array}$$

Example 5.1. No nontrivial implications can be added to the above diagram.

(1) *A U-space which is not a semi-basically disconnected space:* By 6W in [13] $\beta\,\mathbb{N} \setminus \mathbb{N}$ is not a basically disconnected space. Since $\mathbb{N}$ is strongly zero-dimensional, $\beta\,\mathbb{N}$ is also strongly zero-dimensional by Theorem 6.2.12 in [12]. Using Theorem 6.2.4 in [12], it is easy to see that $\beta\,\mathbb{N} \setminus \mathbb{N}$ is also strongly zero-dimensional. On the other hand, $\beta\,\mathbb{N} \setminus \mathbb{N}$ is an F-space by Theorem 14.27 in [13], so it is a U-space by Theorem 2.1, see also [23, Example 9.17]. Now we consider a copy $\mathbb{N}'$ of $\mathbb{N}$ and take $X = \beta\,\mathbb{N} \setminus \mathbb{N}$ and $Y = \beta\,\mathbb{N}' \setminus \mathbb{N}'$. Let W be the free union of X and Y. Since X is not basically disconnected, there exists $f \in C(X)$ such that $\operatorname{cl}_X \operatorname{coz} f$ is not open. If we take $f' \in C(Y)$ as the copy of f, then clearly $\operatorname{cl}_Y \operatorname{coz} f'$ is not open. Now define $h \in C(W)$ as follows:

$$h(x) = \begin{cases} f^2(x) & x \in \operatorname{coz} f \\ -f'^2(x) & x \in \operatorname{coz} f' \\ 0 & x \in Z(f) \cup Z(f'). \end{cases}$$

Clearly, $\mathrm{cl}_W \operatorname{pos} h = \mathrm{cl}_X \operatorname{coz} f$, $\mathrm{cl}_W \operatorname{neg} h = \mathrm{cl}_Y \operatorname{coz} f'$, and non of them is open in W, so W is not a semi-basically disconnected space. On the other hand, since X and Y are strongly zero-dimensional F-spaces, so is W. Hence, W is a U-space.

(2) *A basically disconnected space which is not a semi-P-space:* We consider the space $\beta\,\mathbb{N}$. It is a basically disconnected space by 6M in [13] but it is not a semi-P-space by the argument preceding Theorem 3.6 since $\beta\,\mathbb{N}$ is not finite.

(3) *A basically disconnected semi-P-space which is not a P-space:* The space $\Sigma = \mathbb{N}\cup\{\sigma\}$ in [14, 4M] is basically disconnected and a semi-P-space. However, the space Σ is not a P-space by [14, 4M(4)].

(4) *A semi-P-space which is not a basically disconnected space:* Example 3.2 in [1] introduce a P_F-space which is not basically disconnected. Hence, this space is a semi-P-space which is not basically disconnected.

A space X is called a P'-space if for all $f \in C(X)$, and all $x \in Z(f)$, there is a deleted neighborhood U' of x such that either $f(U') = 0$ or $f(U') > 0$ or $f(U') < 0$, for more details see [14]. For instance, the space Σ is a P'-space, see also Examples 8.5 and 8.6 in [14]. Every P'-space is basically disconnected, see Theorems 8.3 and 8.4 in [14]. At first glance, the two concepts of P'-space and semi-P-space may seem the same, but the next example shows that this is not the case.

Example 5.2. (1) *A semi-P-space which is not a P'-space:* See Example 3.2 in [1].

(2) *A P'-space which is not a semi-P-space:* Let $\Sigma' = -\mathbb{N} \cup \{\sigma'\}$ be a copy of Σ, where $-\mathbb{N} = \{-n : n \in \mathbb{N}\}$. Then clearly, Σ as well as Σ' are P'-spaces. Now we consider the free union $X = \Sigma \dot{\cup} \Sigma'$. It is easy to see that X is a P'-space. Now we show that X is not a semi-P-space. Define $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \frac{1}{x} & x \in \mathbb{N} \cup -\mathbb{N} \\ 0 & x \in \{\sigma, \sigma'\}. \end{cases}$$

Then $f \in C(X)$. Clearly, $\operatorname{pos} f = \mathbb{N}$, $\operatorname{neg} f = -\mathbb{N}$ and none of them is closed. This shows that X is not a semi-P-space.

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References

[1] Abu Osba, E., Henriksen, M. and Alkam, O., 2004. Essential P-spaces: a generalization of door spaces, *Comment. Math. Univ. Carol.*, (45), 509-518.

[2] Al-Ezeh, H., 1989. Exchange PF-rings and almost PP-rings, *Internat. J. Math. Math. Sci.*, (12), 725-727. doi:10.1155/S016117128900089X

[3] Azarpanah, F., 2002. When is $C(X)$ a clean ring?, *Acta Math. Hungar.*, (94), 53-58. doi:10.1023/A:1015654520481

[4] Azarpanah, F., Ghashghaei, E. and Ghoulipour, M., 2020. $C(X)$: Something old and something new, *Commun. Algebra*, (49), 185-206. doi:10.1080/00927872.2020.1797070

[5] Azarpanah, F., Mohamadian, R. and Monjezi, P., 2021. On P_F-spaces, *Topology Appl.*, (302), 107821, 8pp. doi:10.1016/j.topol.2021.107821

[6] Banaschewski, B., 2000. Gelfand and exchange rings: their spectra in pointfree topology, *Arab. J. Sci. Eng. Sect. C: Theme Issues*, (25), 3-22.

[7] Burgess, W.D. and Raphael, R., 2020. Tychonoff spaces and a ring theoretic order on $C(X)$, *Topology Appl.*, (279), 107250, 10 pp. doi:10.1016/j.topol.2020.107250

[8] Brookshear, J.G., 1978. On projective prime ideals in $C(X)$, *Proc. Amer. Math. Soc.*, (69), 203-204. doi:10.1090/S0002-9939-1978-0470929-5

[9] De Marco, G., 1983. Projectivity of pure ideals, *Rend. Sem. Mat. Univ. Padova*, (68), 289-304.

[10] De Marco, G. and Orsatti, A., 1971. Commutative rings in which every prime ideal is contained in a unique maximal ideal, *Proc. Amer. Math. Soc.*, (30), 459-466. doi:10.2307/2037716

[11] Endo, S., 1961. On semi-hereditary rings, *J. Math. Soc. Japan*, (13), 109-119. doi: 10.2969/JMSJ/01320109

[12] Engelking, R., 1989. *General Topology*, Heldermann Verlag, Berlin.

[13] Gillman, L. and Jerison, M., 1960. *Rings of Continuous Functions*, D. Van Nostrand Company, Inc., Princeton, N. J..

[14] Gillman, L. and Henriksen, M., 1956. Rings of continuous functions in which every finitely generated ideal is principal, *Trans. Amer. Math. Soc*, (82), 366-391. doi:10.1090/S0002-9947-1956-0078980-4

[15] Gillman, L. and Kohls, C.W., 1959. Convex and pseudoprime ideals in rings of continuous functions,*Math. Z.*, (72), 399-409. doi:10.1007/BF01162963

[16] Goodearl, K.R., Warfield, R.B., 1976. Algebras over zero-dimensional rings, *Math. Ann.*, (223), 157-168. doi:10.1007/BF01360879

[17] Johnstone, P.T., 1982. *Stone Spaces*, Cambridge Stud. Adv. Math., 3, Cambridge Univ. Press, Cambridge.

[18] Knox, M.L., Levy, R. and McGovern, W.Wm. and Shapiro, J., 2009. Generalizations of complemented rings with applications to rings of functions, *J. Alg. Appl.*, (8), 17-40. doi:10.1142/S021949880903138

[19] Kohls, C.W., 1957. Ideals in rings of continuous functions, *Fund. Math.*, (45), 28-50.

[20] Lam, T.Y., 1999. *Lectures on Modules and Rings, Graduate Texts in Mathematics*, 189. Springer-Verlag, New York.

[21] Mason, G., 1973. z-ideals and prime ideals, *J. Algebra*, (26), 280-297. doi:10.1016/0021-8693(73)90024-0

[22] McGovern, W.Wm., Clean semiprime f-rings with bounded inversion, *Comm. Algebra*, (31), 3295-3304. doi:10.1081/aGB-120022226

[23] McGovern, W.Wm., Puninski, G. and Rothmaler, P., 2007. When every projective module is a direct sum of finitely generated modules, *J. Algebra*, (315), 454-481. doi:10.1016/j/jalgebra.2007.01.043

[24] Nicholson, W.K., 1977. Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, (229), 269-278. doi:10.2307/1998510

[25] Niefeld S. and Rosenthal, K., 1987. Sheaves of integral domains on stone spaces, *J. Pure Appl. Algebra*, (47), 173-179. doi:10.1016/0022-4049(87)90060-0

[26] Varadarajan, K., 2000. Study of Hopficity in certain classes of rings, *Comm. Algebra*, (28), 771-783. doi:10.1080/00927870008826858

[27] Xiang, Y., 2017. A generalization of VNL-Rings and PP-Rings, *Journal of Mathematical Research with Applications*, (37), 199-208.

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