Shahid Chamran University of Ahvaz

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# ON THE LENGTHS AND MULTIPLICITIES OF SOME GRADED GENERALIZED LOCAL COHOMOLOGY MODULES

SAEEDEH KARIMBEYGLU, AHMAD KHOJALI* AND NASER ZAMANI

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Abstract. Let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with a local base ring $(R_0, \mathfrak{m}_0)$ of Krull dimension 2. Let $R_+ = \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of $R$. Let $M$ and $N$ be two finitely generated $\mathbb{Z}$-graded $R$-modules such that, $\mathrm{pd}_R(M)$, the projective dimension of $M$, is finite. Under some conditions on $H^i_{R_+}(M,N)$, it will be shown that there exists $x \in \mathfrak{m}_0$ such that $\ell_{R_0}((0 :_{H^i_{R_+}(M,N)_n} x))$ is of reverse polynomial type of degree less than $i$; i.e., there exists a polynomial $P(X) \in \mathbb{Q}[X]$ of degree less than $i$ such that $\ell_{R_0}((0 :_{H^i_{R_+}(M,N)_n} x)) = P(n)$, for all $n \ll 0$. Also, it is proved that, $\ell_{R_0}(\Gamma_{\mathfrak{m}_0 R} H^i_{R_+}(M, N/xN)_n)$ and $\ell_{R_0}(H^1_{\mathfrak{m}_0 R} H^i_{R_+}(M,N)_n)$, for a parameter element $x \in \mathfrak{m}_0$, are of reverse polynomial type of degree less than $i$. Finally, for an $\mathfrak{m}_0$-primary ideal $\mathfrak{q}_0$, we study the asymptotic behaviour of the multiplicities $e_d(\mathfrak{q}_0, H^i_{R_+}(M,N)_n)$ for $n \ll 0$, where $d = \dim_{\widehat{R}} \big({}^*D(H^i_{R_+}(M,N) \otimes_{R_0} \widehat{R_0})\big)$.

## 1. Introduction

Throughout $R$ is a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0, k_0)$. This means that $R_0$ is a Noetherian local ring and $R = R_0[R_1]$ is a finitely generated algebra over $R_0$. Let $R_+ = \oplus_{n\in\mathbb{N}} R_n$ be the irrelevant ideal of $R$ and $\mathfrak{m} = \mathfrak{m}_0 \oplus R_+$. Then $\mathfrak{m}$ is the only graded maximal ideal of $R$, and so $R$ is an *local ring. Let $M = \oplus_{n\in\mathbb{Z}} M_n$

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*Corresponding author.

and $N = \oplus_{n\in\mathbb{Z}} N_n$ be two finitely generated graded $R$-modules. Following [11], the $i$-th generalized local cohomology module of $M$ and $N$ supported in $R_+$ is defined as $H^i_{R_+}(M,N) := \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/(R_+)^n M, N)$. It is well-known that this module has a natural graded structure $H^i_{R_+}(M,N) = \oplus_{n\in\mathbb{Z}} H^i_{R_+}(M,N)_n$ and that the $R_0$-module $H^i_{R_+}(M,N)_n$ is finitely generated for all $n \in \mathbb{Z}$ and vanishes for all sufficiently large values of $n$, [18, Proposition 2.3]. On the other hand, in general, the asymptotic behavior of the $R_0$-module $H^i_{R_+}(M,N)_n$ for $n \to -\infty$ is mysterious. When $R_0$ is Artinian, then by [19, Theorem 2.2], the $R$-module $H^i_{R_+}(M,N)$ is Artinian. Therefore, by a similar argument as in [13, Theorem 2(ii)], there exists a polynomial $P(X) \in \mathbb{Q}[X]$ such that $P(n) = \ell_{R_0}(H^i_{R_+}(M,N)_n)$ for all $n \ll 0$ and the asymptotic behavior is known. When $\dim(R_0) = 1$, the $R_0$-modules $H^i_{R_+}(M,N)_n$ need not be of finite length, and in this case we showed in [14, Theorem 1.4] that, various graded submodules and quotients of $H^i_{R_+}(M,N)$ are Artinian and an upper bound for the degree of their Hilbert-Kirby polynomials are given. In this paper, we assume that $\dim(R_0) = 2$ and continue our study on the behavior of $H^i_{R_+}(M,N)$, for all $i \in \mathbb{N}_0$. Note that, for $i > \dim R$, by [8, Theorem 3.1], the generalized local cohomology module $H^i_{R_+}(M,N)$ vanishes, and so, without any loss of generality, we may assume that $i \leq \dim R$.

In Section 2, among other things, we prove Proposition 2.1, which is needed to prove the main Theorems **A** and **B**.

In Section 3, under some conditions on the set of associated primes of $H^i_{R_+}(M,N)$ and its homogeneous components, we show that there exists $x \in \mathfrak{m}_0$ such that the length of the $R_0$-modules $(0 :_{H^i_{R_+}(M,N)_n} x)$ is of reverse polynomial type. More precisely, we prove the following theorem.

**Theorem A.** Let $R$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$ of dimension 2. Assume that $\mathrm{pd}_R(M)$ is finite. Let $i$ be a non-negative integer and assume that there exist only finitely many $\mathfrak{p} \in \mathrm{Ass}_R(H^i_{R_+}(M,N))$ such that $\dim(R_0/R_0 \cap \mathfrak{p}) \geq 1$. If $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ for infinitely many $n < 0$, then there exists $x \in \mathfrak{m}_0$ such that $\ell_{R_0}(0 :_{H^i_{R_+}(M,N)_n} x)$ is of reverse polynomial type of degree less than $i$.

In Section 4, we concentrate on the multiplicities of the $R_0$-modules $H^i_{R_+}(M,N)_n$, $n \in \mathbb{Z}$. The main theorem we will prove in this section is:

**Theorem B.** Let $\dim(R_0) = 2$, $\mathfrak{q}_0$ an $\mathfrak{m}_0$-primary ideal of $R_0$ and let $\mathrm{pd}_R(M) < \infty$. If $\dim_{R_0}(H^i_{R_+}(M,N)_n) \geq 1$ for all $n \ll 0$, then $e_d(\mathfrak{q}_0, H^i_{R_+}(M,N)_n)$ is of reverse polynomial type of degree less than $i$ where $d = \dim_{\widehat{R}}\big({}^*D(H^i_{R_+}(M,N) \otimes_{R_0} \widehat{R_0})\big)$.

Here $\widehat{R} = R \otimes_{R_0} \widehat{R_0}$, where $\widehat{R_0}$ is the $\mathfrak{m}_0$-adic completion of $R_0$ and the functor ${}^*D(-)$ will be explained in Remark 4.2(4).

## 2. Preliminaries

In this section, we provide some preliminaries which are needed in the sequel. In what follows, $R$ will be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$ and $M, N$ are assumed to be finitely generated graded $R$-modules. As usual, for a graded ideal $\mathfrak{a}$ of $R$, and for a graded $R$-module $X$, $\Gamma_{\mathfrak{a}}(X) = \bigcup_{n\in\mathbb{N}}(0 :_X \mathfrak{a}^n)$ is the $\mathfrak{a}$-torsion (graded) submodule of $X$.

Recall that, a sequence $\{S_n\}_{n\in\mathbb{Z}}$ of sets, is called asymptotically stable for $n \to -\infty$, if there exists $m \in \mathbb{Z}$ such that $S_n = S_m$ for all $n \leq m$ (see [2]).

**Proposition 2.1.** *Let $(R_0, \mathfrak{m}_0)$ be local, and let $\mathrm{pd}_R(M)$ be finite. Then the following statements hold.*

(1) *$H^i_{R_+}(M,N)_n$ is finitely generated for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, and it is zero for all $i$ and for all large $n$.*
(2) *For each $i \in \mathbb{N}_0$ the $R$-module $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is Artinian.*
(3) *Let $i$ be a non-negative integer, $\dim(R_0) = 2$ and let $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)$, for infinitely many $n < 0$. If $\{\mathfrak{p} \in Ass_R\big(H^i_{R_+}(M,N)\big) | \dim(R_0/\mathfrak{p} \cap R_0) \geq 1\}$ is a finite set, then $\mathfrak{m}_0 \in \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)$ for all $n \ll 0$. Moreover, the sequence $\{\mathrm{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n \to -\infty$.*

*Proof.* For part(1) see [18, Proposition 2.3], part (2) is true by [19, Lemma 3.5], and part (3) is valid [20, Theorem 3.7]. □

**Definition and Remark 2.2.** *Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be an Artinian $R$-module. Then all the $R_0$-modules $M_n$ are Artinian, and $M_n$ is finitely generated $R_0$-module for all $n \ll 0$ if and only if $\ell_{R_0}(M_n) < \infty$ for all $n \ll 0$. Moreover, by a similar argument as in* [13, Theorem 2] *we see that there is a uniquely determined polynomial $P_M(X) \in \mathbb{Q}[X]$ of degree at most $\dim_{k_0}(R_1/\mathfrak{m}_0 R_1) - 1$ such that $\ell_{R_0}(M_n) = P_A(n)$ for all $n \ll 0$. In this case, we say that $M$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial $P_M(X)$.*

To prove Theorems **A** and **B** we need the following proposition. See also [12] Section 4.

**Proposition 2.3.** *Let $i$ be a non-negative integer, $\dim R_0 = 2$ and let $M, N$ be two finitely generated graded $R$-modules such that $\mathrm{pd}_R(M)$ is finite. Then the following statements hold:*

(1) *$H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$.*

(2) *Assume that $x \in \mathfrak{m}_0$ is part of a system of parameter elements for $R_0$. Then $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M, N/xN))$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$.*

(3) *$H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M, N))$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$.*

(4) *$\big(\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)\big) + xH^i_{R_+}(M, N)\big)/xH^i_{R_+}(M, N)$, where $x \in \mathfrak{m}_0$ is both an $N$-regular element and a parameter element for $R_0$, is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$.*

*Proof.* (1) According to Proposition 2.1(2), $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is Artinian. Also, by Proposition 2.1(1), all components of $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ are finitely generated $R_0$-modules. These together give that all components of $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ have finite length. Hence by Definition 2.2, $H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))$ is a Artinian graded $R$-module with Hilbert-Kirby polynomial. We set $p = \mathrm{pd}_R(M)$. Now, by induction on $p$, we are going to prove that $\deg P_{H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N))}(X) < i$. Let $t \in \mathbb{N}$ be such that $\Gamma_{\mathfrak{m}_0 R}(N) = (0 :_N \mathfrak{m}_0^t)$. If $p = 0$ then $M \cong \bigoplus_{i=1}^n R(-a_i)$ for some $n \in \mathbb{N}$ and some natural integers $a_i$. Thus

$$\begin{aligned} H^i_{R_+}\big(M, \Gamma_{\mathfrak{m}_0 R}(N)\big) &\cong \bigoplus_{i=1}^n H^i_{R_+}\big(\Gamma_{\mathfrak{m}_0 R}(N)\big)(a_i) \\ &\cong \bigoplus_{i=1}^n H^i_{R_+(R/\mathfrak{m}_0^t R)}\big(\Gamma_{\mathfrak{m}_0 R}(N)\big)(a_i), \end{aligned}$$

where the last isomorphism holds by the Independence Theorem ([4, Theorem 4.2.1]). Therefore, we may (and do) assume that $R_0$ is Artinian. Note that, by [5, Proposition 1.5.15], a finitely generated graded $R$-module is projective if and only if it is free and, by [4, Theorem 3.4.10], the local cohomology modules $H^i_{R_+}(.)$, commute with direct sum. So, by [4, Theorem and Definition 17.1.11] the result follows for $p = 0$ (note that for

$i=0$, $H^0_{R_+}\big(M,\Gamma_{\mathfrak{m}_0R}(N)\big)=\Gamma_{\mathfrak{m}}\big(\mathrm{Hom}_R(M,N)\big)$ is a finite length $R$-module). So, let $p>0$. Consider the short exact sequence $0\longrightarrow K\longrightarrow F\cong\oplus_{i=1}^m R(-b_i)\longrightarrow M\longrightarrow 0$ (of graded $R$-modules and graded $R$- homomorphisms), where $m,b_i,1\leq i\leq m$ are natural integers, which leads into the following exact sequence

$$H^{i-1}_{R_+}(K,\Gamma_{\mathfrak{m}_0R}(N))\xrightarrow{\alpha}H^i_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N))\xrightarrow{\beta}H^i_{R_+}(F,\Gamma_{\mathfrak{m}_0R}(N)).$$

By additivity of $\ell_{R_0}(.)$ and the induction hypothesis, we deduce that, for all $n\ll 0$,

$$\begin{aligned}
P_{H^i_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N))}(n)&=\ell_{R_0}\big(H^i_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N))_n\big)\\
&=\ell_{R_0}\big(\mathrm{im}(\alpha)_n\big)+\ell_{R_0}\big(\mathrm{im}(\beta)_n\big)\\
&\leq\ell_{R_0}\big(H^{i-1}_{R_+}(K,\Gamma_{\mathfrak{m}_0R}(N))_n\big)+\ell_{R_0}\big(H^i_{R_+}(F,\Gamma_{\mathfrak{m}_0R}(N))_n\big)\\
&=\ell_{R_0}\big(H^{i-1}_{R_+}(K,\Gamma_{\mathfrak{m}_0R}(N))_n\big)+m\ell_{R_0}\big(H^i_{R_+}(\Gamma_{\mathfrak{m}_0R}(N))_n\big)\\
&=P_{H^{i-1}_{R_+}(K,\Gamma_{\mathfrak{m}_0R}(N))}(n)+mP_{H^i_{R_+}(\Gamma_{\mathfrak{m}_0R}(N))}(n).
\end{aligned}$$

Thinking of the fact that, $P_{H^{i-1}_{R_+}(K,\Gamma_{\mathfrak{m}_0R}(N))}(X)$ and $P_{H^i_{R_+}(\Gamma_{\mathfrak{m}_0R}(N))}(X)$, by induction hypothesis and the case $p=0$, are polynomials of degree less than $i-1$ and $i$, respectively, then, $\deg P_{H^i_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N))}(X)<i$, as desired.

(2) First, we prove that $H^i_{R_+}(M,N/xN)\cong H^i_{Rx+R_+}(M,N/xN)$. Let $\mathbf{J}$ be the minimal injective resolution of $N/xN$. Note that, by [4, Corollary 2.1.6], one may assume that $\Gamma_{Rx}(\mathbf{J})=\mathbf{J}$, and so, $\Gamma_{R_+}(\mathbf{J})=\Gamma_{Rx+R_+}(\mathbf{J})$. Now, by [17, Proposition 2.1] or [7, Lemma 2.1], we have

$$\begin{aligned}
H^i_{R_+}(M,N/xN)&\cong H^i(\Gamma_{R_+}(\mathrm{Hom}_R(M,\mathbf{J})))\\
&\cong H^i(\mathrm{Hom}_R(M,\Gamma_{R_+}(\mathbf{J})))\\
&\cong H^i(\mathrm{Hom}_R(M,\Gamma_{Rx+R_+}(\mathbf{J})))\\
&\cong H^i(\Gamma_{Rx+R_+}(\mathrm{Hom}_R(M,\mathbf{J})))\\
&\cong H^i_{Rx+R_+}(M,N/xN)
\end{aligned}$$

as desired. Now, let $y$ be such that $x,y$ is a system of parameter for $R_0$; i.e. $\sqrt{(x,y)}=\mathfrak{m}_0$. Therefore, by [9, Lemma 3.1], we have the exact sequence

$$H^i_{Ry+Rx+R_+}(M,N/xN)\longrightarrow H^i_{Rx+R_+}(M,N/xN)\longrightarrow H^i_{Rx+R_+}(M,N/xN)_y$$

that gives rise into the exact sequence

$$H^i_{Ry+Rx+R_+}(M,N/xN) \longrightarrow \Gamma_{Ry}\big(H^i_{Rx+R_+}(M,N/xN)\big) \longrightarrow 0. \tag{2.1}$$

As $\sqrt{(x,y)} = \mathfrak{m}_0$, then there exists a natural number $n$ such that $\mathfrak{m}_0^n \subseteq (x,y)$, and so, $\sqrt{\mathfrak{m}_0 R} = \sqrt{(x,y)R} = \mathfrak{m}_0 R$. Therefore, $H^i_{Ry+Rx+R_+}(M,N/xN) \cong H^i_{\mathfrak{m}}(M,N/xN)$ is Artinian, where, by considering the exact sequence (2.1) and the fact that

$$\begin{aligned}\Gamma_{Ry}\big(H^i_{Rx+R_+}(M,N/xN)\big) &\cong \Gamma_{Ry+Rx}\big(H^i_{Rx+R_+}(M,N/xN)\big)\\ &\cong \Gamma_{\mathfrak{m}_0 R}\big(H^i_{Rx+R_+}(M,N/xN)\big)\end{aligned}$$

we deduce that $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N/xN)\big)$ is an Artinian graded $R$-module. To complete the proof, we prove, by induction on $p = \mathrm{pd}_R(M)$, that the degree of its Hilbert-Kirby polynomial is less than $i$. In the case that $p = 0$, then $M \cong \bigoplus_{i=1}^m R(-a_i)$ where $m \in \mathbb{N}$ and $a_i$ are some natural integers. Then $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N/xN)\big) \cong \bigoplus_{i=1}^m \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(N/xN)\big)(a_i)$ and the result follows by [1, Theorem 3.5]. Now suppose, inductively, that $p > 0$ and the result has been proved for all $R$-modules of projective dimension less than $p$. There exist a graded free $R$-module $F \cong \oplus_{i=1}^r R(-b_i)$, where $r$ and $b_i, 1 \leq i \leq r$ are some natural integers, and a short exact sequence (of graded $R$-modules and $R$-homomorphisms) $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ from which we get the long exact sequence

$$\cdots \longrightarrow H^{i-1}_{R_+}(K,N/xN) \xrightarrow{\alpha} H^i_{R_+}(M,N/xN) \xrightarrow{\beta} H^i_{R_+}(F,N/xN) \longrightarrow \cdots. \tag{2.2}$$

By induction hypothesis and, by what we have proved for $p = 0$, one deduces that $\Gamma_{\mathfrak{m}_0 R}(H^{i-1}_{R_+}(K,N/xN))$ and $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(F,N/xN)) \cong \oplus_{i=1}^r \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(N/xN))(b_i)$ are Artinian graded $R$-modules with Hilbert-Kirby polynomials of degrees less than $i-1$ and $i$, respectively. Now, the short exact sequence (2.2) gives the short exact sequences $0 \longrightarrow \mathrm{im}(\alpha) \longrightarrow H^i_{R_+}(M,N/xN) \longrightarrow \mathrm{im}(\beta) \longrightarrow 0$ and from left exactness of $\Gamma_{\mathfrak{m}_0 R}(\cdot)$ we get the short exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(\mathrm{im}(\alpha)) \longrightarrow \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N/xN)) \longrightarrow \Gamma_{\mathfrak{m}_0 R}(\mathrm{im}(\beta))$$

at which, additivity of the length function, implies that $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N/xN))$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$.

(3) First of all note that finiteness of $N$ implies finiteness of Bass numbers of $N$, and so, by [7, Lemma 2.1] and an inductive argument on $\dim(N)$, one may deduce that

$H^i_{\mathfrak{m}}(M,N)$, for all $i \in \mathbb{N}_0$, is Artinian. There exist $x, y \in \mathfrak{m}_0$ such that $\sqrt{(x,y)R_0} = \mathfrak{m}_0$. By [4, Proposition 8.1.2], we have the following short exact sequence

$$0 \longrightarrow H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N))) \longrightarrow H^1_{(x,y)R}(H^i_{R_+}(M,N)) \longrightarrow \Gamma_{Rx}(H^1_{Ry}(H^i_{R_+}(M,N))) \longrightarrow 0. \tag{2.3}$$

So, it is enough to show that $H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N)))$ and $\Gamma_{Rx}(H^1_{Ry}(H^i_{R_+}(M,N)))$ are Artinian $R$-modules. Now, by [9, Lemma 3.1], we get the long exact sequence

$$H^{i-1}_{R_+}(M,N) \xrightarrow{\nu} H^{i-1}_{R_+}(M,N)_y \longrightarrow H^i_{(y,R_+)}(M,N) \longrightarrow H^i_{R_+}(M,N) \xrightarrow{\eta} H^i_{R_+}(M,N)_y, \tag{2.4}$$

of graded $R$-modules and $R$-homomorphisms. Note that, by [4, Proposition 8.1.2] we have $\operatorname{coker}(\nu) = H^1_{Ry}(H^{i-1}_{R_+}(M,N))$ and $\ker(\eta) = \Gamma_{Ry}(H^i_{R_+}(M,N))$, and hence, from the long exact sequence (2.4), we get the short exact sequence

$$0 \longrightarrow H^1_{Ry}(H^{i-1}_{R_+}(M,N)) \longrightarrow H^i_{(y,R_+)}(M,N) \longrightarrow \Gamma_{Ry}(H^i_{R_+}(M,N)) \longrightarrow 0. \tag{2.5}$$

By applying $\Gamma_{Rx}(\cdot)$ to the short exact sequence (2.5) and considering the fact that the arithmetic rank of the ideal $Rx$ is one, that guarantees vanishing of $H^2_{Rx}\big(H^1_{Ry}(H^i_{R_+}(M,N))\big)$, we get the long exact sequence

$$\cdots \longrightarrow H^1_{Rx}(H^i_{(y,R_+)}(M,N)) \longrightarrow H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N))) \longrightarrow 0 \tag{2.6}$$

which in turn implies that $H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N)))$ is an Artinian graded $R$-module, provided that $H^1_{Rx}(H^i_{(y,R_+)}(M,N))$ is so.

From the short exact sequence (2.5), by using $x$ and $(Ry, R_+)$ instead of $y$ and $R_+$, respectively, one may get the short exact sequence

$$0 \longrightarrow H^1_{Rx}(H^i_{(y,R_+)}(M,N)) \longrightarrow H^{i+1}_{(x,y,R+)}(M,N) \longrightarrow \Gamma_{Rx}\big(H^{i+1}_{(y,R_+)}(M,N)\big) \longrightarrow 0 \tag{2.7}$$

which implies that $H^1_{Rx}(H^i_{(y,R_+)}(M,N))$ and $\Gamma_{Rx}\big(H^{i+1}_{(y,R_+)}(M,N)$ are Artinian. Consequently, by applying $\Gamma_{Rx}(\cdot)$ to the short exact sequence (2.5), we get the monomorphism

$$0 \longrightarrow \Gamma_{Rx}\big(H^1_{Ry}(H^i_{R_+}(M,N))\big) \longrightarrow \Gamma_{Rx}\big(H^{i+1}_{(y,R_+)}(M,N)\big) \tag{2.8}$$

which means that $\Gamma_{Rx}\big(H^1_{Ry}(H^i_{R_+}(M,N))\big)$ is Artinian, too. By a similar reasoning, as it was done in the proof of part (1), it is easily seen that $H^i_{\mathfrak{m}}(M,N)$ is an Artinian

graded $R$-module with Hilbert-Kirby polynomial of degree less that $i-1$. This, by using the exact sequences (2.6), (2.7) and (2.8) implies that $H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N)))$ and $\Gamma_{Rx}(H^1_{Ry}(H^i_{R_+}(M,N)))$ are Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$. Thus $T := H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)) \cong H^1_{(x,y)R}(H^i_{R_+}(M,N))$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial. Now, by additivity of the length function, and the short exact sequence (2.3), for all $n \ll 0$, we have

$$\begin{aligned} P_T(n) &= \ell_{R_0}\big(H^1_{(x,y)R}(H^i_{R_+}(M,N))_n\big) \\ &= \ell_{R_0}\big(H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N)))_n\big) + \ell_{R_0}\big(\Gamma_{Rx}(H^1_{Ry}(H^i_{R_+}(M,N)))_n\big) \\ &= P_{H^1_{Rx}(\Gamma_{Ry}(H^i_{R_+}(M,N)))}(n) + P_{\Gamma_{Rx}(H^1_{Ry}(H^i_{R_+}(M,N)))}(n) \end{aligned}$$

which implies that $\deg P_{H^1_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N))}(X) < i$, as desired.

(4) The short exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ implies the long exact sequence

$$\cdots \longrightarrow H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/xN) \longrightarrow \cdots,$$

from which we get the exact sequence

$$0 \longrightarrow H^i_{R_+}(M,N)/xH^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/xN). \tag{2.9}$$

Applying $\Gamma_{\mathfrak{m}_0 R}(\cdot)$ to the exact sequence (2.9), gives the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\big) \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N/xN)\big). \tag{2.10}$$

Thinking of the fact that $\big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)) + xH^i_{R_+}(M,N)\big)/xH^i_{R_+}(M,N)$ is a submodule of $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\big)$, and Artinianness of the later, which is proved in part (2), the exact sequence (2.10), and Proposition 2.1(1), one may deduce that $S := \big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)) + xH^i_{R_+}(M,N)\big)/xH^i_{R_+}(M,N)$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial. Now, in view of the exact sequence (2.10), for all $n \ll 0$, we have

$$\begin{aligned} P_S(n) &= \ell_{R_0}\Big(\big(\Gamma_{\mathfrak{m}_0}\big(H^i_{R_+}(M,N)_n\big) + xH^i_{R_+}(M,N)_n\big)/xH^i_{R_+}(M,N)_n\Big) \\ &\leq \ell_{R_0}\Big(\Gamma_{\mathfrak{m}_0}\big(H^i_{R_+}(M,N)_n/xH^i_{R_+}(M,N)_n\big)\Big) \\ &= \ell_{R_0}\Big(\Gamma_{\mathfrak{m}_0}\big(H^i_{R_+}(M,N/xN)_n\big)\Big) \\ &\leq P_{\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M,N/xN)\right)}(n). \end{aligned}$$

Therefore, $P_S(X)$ is a polynomial of degree less than $i$, as desired. □

**Remark 2.4.** (1) Let $R = \oplus_{i\in\mathbb{N}_0} R_i$ be a graded homogeneous *local ring and let $N = \oplus_{i\in\mathbb{Z}} N_i$ be a finitely generated graded $R$-module. According to [15, Lemma 1.1.2] the function $\psi : \operatorname{Ass}_R(N) \longrightarrow \operatorname{Ass}_{R_0}(N)$ defined by $\psi(\mathfrak{p}) = \mathfrak{p} \cap R_0$ is surjective and so $\operatorname{Ass}_{R_0}(N) = \bigcup_{i\in\mathbb{Z}} \operatorname{Ass}_{R_0}(N_i)$ is a finite set. Since $N$ is not assumed to be a finitely generated $R_0$-module then the notation $\operatorname{depth}(\mathfrak{m}_0, N)$, to be meaningful, needs to be investigated. It is easily seen that an element $x \in \mathfrak{m}_0$ is $N$-regular if and only if it is $N_i$-regular for all $i \in \mathbb{Z}$ with $N_i \neq 0$. Let $\mathbf{x}$ and $\mathbf{y}$ be two maximal $N$-regular sequences (in $\mathfrak{m}_0$) of length $n$ and $m$, respectively. Then, by the prime avoidance theorem, there is $i \in \mathbb{Z}$ such that $\mathfrak{m}_0 \in \operatorname{Ass}_{R_0}(N_i/\mathbf{x}N_i)$. Since $\mathbf{y}$ is also an $N_i$-regular sequence, then $n \leq m$, and so, $n = m$ by symmetry.

(2) Let $L$ be an $R$-module and assume that $\mathbf{x}$ is an $L$-regular sequence such that $\operatorname{pd}_R(L/\mathbf{x}L)$ is finite. Then, it is easily seen, that, $\operatorname{pd}_R(L/\mathbf{x}^nL)$ is finite, for all $n \in \mathbb{N}$. In fact, by [5, Theorem 1.1.8], $\mathbf{x}^{n-1}L/\mathbf{x}^nL \cong L/\mathbf{x}L$. Now, for any natural integer $n$, from the short exact sequence $0 \to \mathbf{x}^{n-1}L/\mathbf{x}^nL \to L/\mathbf{x}^nL \to L/\mathbf{x}^{n-1}L$ and an inductive argument on $n$ the result follows.

**Corollary 2.5.** *Let $i \in \mathbb{N}_0$, $\dim(R_0) = 2$ and assume that $\mathbf{x} = x_1, x_2 \in \mathfrak{m}_0$ is a sequence of elements which is $R$-regular and that $x_1$ is $N$-regular. Then, for each non-negative integer $n$, the following statements hold.*

(1) *The $R$-modules $\Big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(R/\mathbf{x}^nR, N)) + x_1H^i_{R_+}(R/\mathbf{x}^nR, N)\Big)/x_1H^i_{R_+}(R/\mathbf{x}^nR, N)$ and $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(R/\mathbf{x}^nR, N/x_1N)\big), H^i_{R_+}\big(R/\mathbf{x}^nR, \Gamma_{\mathfrak{m}_0 R}(N)\big)$ are graded Artinian with Hilbert-Kirby polynomials of degree less than $i$.*

(2) *If $\mathbf{x}$ is also an $N$-regular sequence, $J = \langle x_2\rangle$ and $\operatorname{pd}_R(M)$ is finite (e.g. $R$ is regular) then one may replace $R/\mathbf{x}^nR$ with $M/J^nM$ and the statements of* (1) *are still true.*

*Proof.* Note that it is possible to find $x_1$ in a way that to be regular on both of $R$ and $N$ provided that $\operatorname{depth}(\mathfrak{m}_0, N)$ is a positive integer. Now, by Remark 2.4(2), we know that $\operatorname{pd}_R(R/\mathbf{x}^nR) \leq 2$. So, the result is obvious by Proposition 2.3. Finally, by [5, Theorem 1.1.7], $x_1$ is an $N/J^nN$-regular element and again by Proposition 2.3 we are done. □

**Corollary 2.6.** *Let $R$ be a (graded) regular local ring, $i$ a non-negative integer, and let $\dim(R_0) = 2$ (e.g. $R = \mathbb{F}[[X_1, X_2]][X_3, \ldots, X_n]$ where $X_i$ are indeterminate over*

*the field* $\mathbb{F}$*). Assume that* $\operatorname{depth}(\mathfrak{m}_0, N)$ *is non-zero. Then there exists an element* $x \in \mathfrak{m}_0$ *such that* $\big(\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)\big)+xH^i_{R_+}(M,N)\big)/xH^i_{R_+}(M,N), \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N/xN)\big)$ *and* $H^i_{R_+}\big(M, \Gamma_{\mathfrak{m}_0 R}(N)\big)$ *are Artinian graded* $R$*-modules with Hilbert-Kirby polynomials of degree less than* $i$*.*

*Proof.* Since $R$ is regular, by the Theorem of Auslander-Buchsbaum-Serre ([5, Theorem 2.2.7]), every $R$-module has finite projective dimension. A prime avoidance argument, gives an element $x \in \mathfrak{m}_0 \setminus \bigcup \operatorname{Ass}_R(N) \cup \bigcup \operatorname{Ass}(R_0)$ which is part of a system of parameters for $R_0$ and, at the same time, an $N$-regular element. Now, the result follows by Proposition 2.3. □

## 3. **Proof of theorem A**

To prove the theorem we need a definition and some facts.

**Definition 3.1.** (1) Let $T$ be an $R_0$-module. An element $x \in \mathfrak{m}_0$ is called $\mathfrak{m}_0$-filter regular with respect to $T$ if $x \notin \bigcup \operatorname{Ass}_{R_0}(T) \setminus \mathfrak{m}_0$. This is equivalent to say that $x$ is a $T/\Gamma_{\mathfrak{m}_0}(T)$-regular element.

Note that if $T$ is finitely generated, then each $\mathfrak{m}_0$-filter regular element is a parameter element with respect to $T$.

(2) A numerical function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is of reverse polynomial type of degree $d$ if there exists a polynomial $P(X) \in \mathbb{Q}[X]$ of degree $d$ such that $f(n) = P(n)$, for all $n \ll 0$.

**Theorem A.** (cf. [3, Proposition 5.4]) Let $R$ be a Noetherian homogeneous ring with base ring $(R_0, \mathfrak{m}_0)$ of dimension 2 and let $\operatorname{pd}_R(M)$ be finite. Let $i \in \mathbb{N}_0$ and assume that there exists only finitely many $\mathfrak{p} \in \operatorname{Ass}_R(H^i_{R_+}(M,N))$ such that $\dim(R_0/R_0 \cap \mathfrak{p}) \geq 1$. If $\mathfrak{m}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ for infinitely many $n < 0$, then there exists $x \in \mathfrak{m}_0$ such that $\ell_{R_0}(0 :_{H^i_{R_+}(M,N)_n} x)$ is of reverse polynomial type of degree less than $i$.

*Proof.* By Proposition 2.1(3), the sequence $\{\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)\}_{n\in\mathbb{Z}}$, is asymptotically stable for $n \to -\infty$, and so, there exists $k_0 \in \mathbb{Z}$ such that, for all $k \leq k_0$ we have $\operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_{k_0}) = \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_k)$. Thus, by assumption, we deduce that $\mathfrak{m}_0 \in \operatorname{Ass}_{R_0}(H^i_{R_+}(M,N)_n)$ for all $n \ll 0$. A prime avoidance argument implies that there exists

$$x \in \mathfrak{m}_0 \setminus \Big( \bigcup_{\mathfrak{p}\in \operatorname{Ass}_R(N/\Gamma_{\mathfrak{m}_0 R(N)})} \mathfrak{p} \cap R_0 \Big) \cup \Big( \bigcup \operatorname{Min}(R_0) \Big) \cup \Big( \bigcup_{j\in\{i,i+1\},\, n\ll 0} \operatorname{Ass}_{R_0}(H^j_{R_+}(M,N)_n)) \setminus \{\mathfrak{m}_0\} \Big).$$

Such an element is a parameter element for $R_0$ and, at the same time, is $\mathfrak{m}_0$-filter regular with respect to $N$ and $H^j_{R_+}(M,N)_n$ for $j \in \{i, i+1\}$ and for all $n \ll 0$. The short exact sequence $0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}(N) \longrightarrow N \longrightarrow \bar{N} = N/\Gamma_{\mathfrak{m}_0 R}(N) \longrightarrow 0$ induces the long exact sequence

$$H^i_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N)) \longrightarrow H^i_{R_+}(M,N) \xrightarrow{\mu} H^i_{R_+}(M,\bar{N}) \longrightarrow H^{i+1}_{R_+}(M, \Gamma_{\mathfrak{m}_0 R}(N)). \tag{3.1}$$

By Proposition 2.1(2), the outer modules in the exact sequence (3.1) are Artinian graded $R$-modules with Hilbert-Kirby polynomial, and so $\ker(\mu)$ and $\operatorname{coker}(\mu)$ are Artinian graded $R$-modules with Hilbert-Kirby polynomial. From the exact sequence (3.1) we get the short exact sequences

$$0 \longrightarrow \ker(\mu) \longrightarrow H^i_{R_+}(M,N) \longrightarrow \operatorname{im}(\mu) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im}(\mu) \longrightarrow H^i_{R_+}(M,\bar{N}) \longrightarrow \operatorname{coker}(\mu) \longrightarrow 0$$

which, in turn, by applying $\operatorname{Hom}_R(R/xR, \cdot)$, imply the exact sequences

$$0 \longrightarrow (0 :_{\ker(\mu)} x) \longrightarrow (0 :_{H^i_{R_+}(M,N)} x) \longrightarrow (0 :_{\operatorname{im}(\mu)} x) \longrightarrow \operatorname{Ext}^1_R(R/xR, \ker(\mu)) \tag{3.2}$$

and

$$0 \longrightarrow (0 :_{\operatorname{im}(\mu)} x) \longrightarrow (0 :_{H^i_{R_+}(M,\bar{N})} x) \longrightarrow (0 :_{\operatorname{coker}(\mu)} x), \tag{3.3}$$

respectively. Again, by Definition 2.2, we deduce that $(0 :_{\ker(\mu)} x)$, $(0 :_{\operatorname{coker}(\mu)} x)$ and the extension module $\operatorname{Ext}^1_R(R/xR, \ker(\mu))$ are Artinian graded $R$-modules with Hilbert-Kirby polynomial of degree less than $i$ and $i+1$, respectively. Thus, to show the claim, it suffices, by the exact sequences (3.2) and (3.3), to prove that $\ell_{R_0}(0 :_{H^i_{R_+}(M,\bar{N})_n} x)$ is of reverse polynomial type of degree less than $i$. Therefore, by replacing $\bar{N}$ with $N$, without loss of generality, we may (and do) assume that $N$ is $\mathfrak{m}_0 R$-torsion-free which implies that $x$ is an $N$-regular element. Now, the short exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ implies the long exact sequence

$$\cdots \longrightarrow H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M, N/xN) \longrightarrow \cdots, \tag{3.4}$$

from which we get the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\big) \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N/xN)\big). \tag{3.5}$$

By Proposition 2.3(2), $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N/xN)\big)$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$, and so, by the exact sequence (3.5) the same is true for $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)/xH^i_{R_+}(M, N)\big)$. Also, from the short exact sequence

$$0 \longrightarrow xH^i_{R_+}(M, N) \longrightarrow H^i_{R_+}(M, N) \longrightarrow H^i_{R_+}(M, N)/xH^i_{R_+}(M, N) \longrightarrow 0$$

(that gives rises by the exact sequence (3.4)), we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(xH^i_{R_+}(M, N)\big) \xrightarrow{\nu} \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)\big) \\ \longrightarrow \Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M, N)/xH^i_{R_+}(M, N)\big). \end{aligned} \tag{3.6}$$

Therefore, from the exact sequence (3.6), we conclude that $\operatorname{coker}(\nu)$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$. Since $x$ is an $\mathfrak{m}_0$-filter regular element with respect to $H^i_{R_+}(M, N)_n$, applying $\Gamma_{\mathfrak{m}_0}(\cdot)$ to the short exact sequence

$$0 \longrightarrow (0 :_{H^i_{R_+}(M,N)_n} x) \longrightarrow H^i_{R_+}(M, N)_n \longrightarrow xH^i_{R_+}(M, N)_n \longrightarrow 0 \tag{3.7}$$

(that results from the exact sequence (3.4)), and using [4, Corollary 2.1.7(i)], we get the short exact sequence

$$0 \longrightarrow (0 :_{H^i_{R_+}(M,N)_n} x) \longrightarrow \Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M, N)_n) \longrightarrow \Gamma_{\mathfrak{m}_0}(xH^i_{R_+}(M, N)_n) \longrightarrow 0, \tag{3.8}$$

for all $n \ll 0$. Now, according to Proposition 2.1(1) and the exact sequences (3.6), (3.7) and (3.8), for all $n \ll 0$, we have

$$\begin{aligned} \ell_{R_0}\big(0 :_{H^i_{R_+}(M,N)_n} x\big) &= \ell_{R_0}\big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M, N))_n\big) - \ell_{R_0}\big(\Gamma_{\mathfrak{m}_0 R}(xH^i_{R_+}(M, N))_n\big) \\ &= \ell_{R_0}\big(\operatorname{coker}(\nu)_n\big) \\ &= P_{\operatorname{coker}(\nu)}(n). \end{aligned}$$

Therefore, $\ell_{R_0}(0 :_{H^i_{R_+}(M,N)_n} x)$ is of reverse polynomial type of degree less than $i$. □

**Remark 3.2.** Let $\dim(R_0) = 2$ and let $\mathfrak{q}_0$ be an $\mathfrak{m}_0$-primary ideal of $R_0$. It is easily seen that for a finitely generated $R_0$-module $T$ we have $\ell_{R_0}(T) = \ell_{\widehat{R_0}}(T \otimes_{R_0} \widehat{R_0})$ and $\mathfrak{q}_0\widehat{R_0}$ is $\mathfrak{m}_0\widehat{R_0}$-primary. Also, by the Flat Base Change Theorem for generalized local cohomology and [10, Lemma 3.2.4], we have

$$(0 :_{H^i_{R_+}(M,\,N)} \mathfrak{q}_0) \otimes_{R_0} \widehat{R_0} \cong (0 :_{H^i_{\hat{R}_+}(M\otimes_{R_0}\widehat{R_0},\, N\otimes_{R_0}\widehat{R_0})} \mathfrak{q}_0\widehat{R_0}),$$

where $\hat{R}_+ = \widehat{R_0} \otimes_{R_0} R_+$. In what follows, we are going to prove that there exists an element $x \in \mathfrak{m}_0$ such that $\ell_{R_0}(0 :_{H^i_{R_+}(M,N)_n} x)$ is of reverse polynomial type of degree less than $i$. By what we have said above, without loss of generality, we may (and do) assume that $(R_0, \mathfrak{m}_0)$ is a complete local ring. For all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, by countable prime avoidance [16, Corollary 2.2], we have

$$\big\{\mathfrak{p} \cap R_0 : \mathfrak{p} \in \mathrm{Ass}_R(N/\Gamma_{\mathfrak{m}_0 R}(N))\big\} \cup \mathrm{Ass}(R_0) \cup \mathrm{Ass}_{R_0}\big(H^i_{R_+}(M,N)_n\big)\big) \setminus \{\mathfrak{m}_0\} \neq \phi.$$

This means that there exists

$$x \in \mathfrak{m}_0 \setminus \big(\bigcup \mathrm{Ass}(R_0)\big) \cup \big(\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(N/\Gamma_{\mathfrak{m}_0 R}(N))} \mathfrak{p} \cap R_0\big) \cup \big(\bigcup_{i \in \mathbb{N}_0,\, n \in \mathbb{Z}} \mathrm{Ass}_{R_0}(H^i_{R_+}(M,\, N)_n)\big)$$

which is a parameter element for $R_0$ and at the same time is an $\mathfrak{m}_0$-filter regular element with respect to $N$ and $H^i_{R_+}(M,N)_n$ for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{N}_0$. Now, as we did in the proof of Theorem **A** it is easily seen that $\ell_{R_0}\big(0 :_{H^i_{R_+}(M,\,N)} x\big)$ is of reverse polynomial type of degree less than $i$.

## 4. **Proof of theorem B**

To prove Theorem **B** we need the following preliminaries.

**Definition 4.1.** Following [2], a graded R-module $X = \oplus_{n\in\mathbb{Z}} X_n$ is said to be tame if $X_n = 0$ for all $n \ll 0$ or $X_n \neq 0$ for all $n \ll 0$.

It is known that any Noetherian or Artinian $R$-module is tame ([2]).

**Remark 4.2.** (1) Recall the following facts ([5, Lemma 4.1.2, Theorem 4.1.3. and Theorem 4.1.4]) :

(*A*) A numerical function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is of polynomial type of degree $d$, if there exists a polynomial $P(X) \in \mathbb{Q}[X]$, of degree $d$, such that $f(n) = P(n)$, for all $n \gg 0$. It is not hard to see that, a numerical function $f$, is of polynomial type of degree $d$, if and only if $\Delta f$ is of polynomial type of degree $d-1$, where $\Delta f(n) = f(n+1) - f(n)$.

(*B*) Let $P(X) \in \mathbb{Q}[X]$ be a polynomial of degree $d$. Then $P(n) \in \mathbb{Z}$ for all $n \gg 0$ if and only if $P(X) = \sum_{i=0}^{d} a_i \binom{X+i}{i}$ for some integers $a_0, \ldots, a_d \in \mathbb{Z}$.

(*C*) Let $R = \oplus_{i\in\mathbb{N}_0} R_i$ be a homogeneous Noetherian ring with Artinian base ring $R_0$. If $L = \oplus_{i\in\mathbb{Z}} L_i$ is a finitely generated graded $R$-module of dimension $d$, then $\ell_{R_0}(L_n)$ is of polynomial type of degree $d-1$.

(2) Let $(S,\mathfrak{n})$ be a local ring, $L$ a finitely generated $S$-module of dimension $d$ and let $\mathfrak{q}$ be an $\mathfrak{n}$-primary ideal of $S$. As $\ell_S(\mathfrak{q}^nL/\mathfrak{q}^{n+1}L)=\ell_S(L/\mathfrak{q}^{n+1}L)-\ell_S(L/\mathfrak{q}^nL)$ then, by what we have said above, the numerical function $\ell_S(L/\mathfrak{q}^nL)$ is of polynomial type of degree $d$. Thus there exists integers $e_0,\ldots,e_d\in\mathbb{Z}$ such that $\ell_S(L/\mathfrak{q}^nL)=\sum_{i=0}^{d}e_i\binom{n+i}{i}$, for all $n\gg 0$. The polynomial $P(X)=\sum_{i=0}^{d}e_i\binom{X+i}{i}$ and the integer $e_d(\mathfrak{q},L):=d!e_d$ are called the *Hilbert-Samuel polynomial* and the *Samuel multiplicity* of $L$, respectively. It is easily seen that $e_d(\mathfrak{q}\widehat{S},\widehat{S}\otimes_S L)=e_d(\mathfrak{q},L)$.

(3) With the same notation as in (2) and according to the *associativity formula for Samuel multiplicity* (see for example [6, Theorem 1] and [5, Corollary 4.7.8]),

$$e_d(\mathfrak{q},L)=\sum_{\mathfrak{p}\in\operatorname{Assh}(L)}\ell_{S_\mathfrak{p}}(L_\mathfrak{p})e_d(\mathfrak{q},S/\mathfrak{p}).$$

where $\operatorname{Assh}(L)$ stands for the set of all $\mathfrak{p}\in\operatorname{Supp}(L)$ such that $\dim(R/\mathfrak{p})=\dim(L)$.

(4) Let $R=\oplus_{i\in\mathbb{N}_0}R_i$ be a homogeneous Noetherian ring, whose base ring $(R_0,\mathfrak{m}_0)$, is complete with respect to the $\mathfrak{m}_0$-adic topology. If $A=\oplus_{n\in\mathbb{Z}}A_n$ is an Artinian graded $R$-module, then ${}^*D(A)={}^*\operatorname{Hom}_R(A,{}^*E_R(R/\mathfrak{m}))$ is a Noetherian graded $R$-module and $P_A(X)=\tilde{P}_{{}^*D(A)}(-X)$, where $\tilde{P}_{{}^*D(A)}(-X)$ stands for the Hilbert polynomial of ${}^*D(A)$(see [14, Remark 2(D)]). Hence, if $\ell_{R_0}(A_n)$ is finite for all $n\in\mathbb{Z}$, then, by [5, Theorem, 4.1.3],

$$\deg P_A(X)=\deg\tilde{P}_{{}^*D(A)}(-X)=\dim\big({}^*D(A)\big)-1. \tag{4.1}$$

Now, we are ready to prove Theorem **B**.

**Theorem B.** (cf. [3, Theorem 5.7]) *Let* $\dim(R_0)=2$, $\mathfrak{q}_0$ *an* $\mathfrak{m}_0$*-primary ideal of* $R_0$ *and suppose that* $\operatorname{pd}_R(M)$ *is finite. If* $\dim_{R_0}(H^i_{R_+}(M,N)_n)\geq 1$ *for all* $n\ll 0$, *then* $e_d(\mathfrak{q}_0,H^i_{R_+}(M,N)_n)$ *is of reverse polynomial type of degree less than* $i$, *where* $d=\dim_{\widehat{R}}\big({}^*D(H^i_{R_+}(M,N)\otimes_{R_0}\widehat{R_0})\big)$.

*Proof.* First of all assume that $\dim_{R_0}(H^i_{R_+}(M,N)_n)=2$, for infinitely many $n$. So, there exists $\mathfrak{p}\in\operatorname{Assh}(H^i_{R_+}(M,N)_n)\cap\operatorname{Assh}(R_0)$, for infinitely many $n$. As $R_\mathfrak{p}$ is a graded ring with an Artinian local base ring, by [7, Lemma 2.1(i)] and [19, Theorem 2.2], we deduce that $H^i_{(R_+)_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})_n\cong H^i_\mathfrak{n}(M_\mathfrak{p},N_\mathfrak{p})_n$ is Artinian, and so, tame, where $\mathfrak{n}$ denotes the ${}^*$maximal ideal of $R_\mathfrak{p}$. It means that $\mathfrak{p}\in\operatorname{Assh}(H^i_{R_+}(M,N)_n)$ for all $n\ll 0$, i.e., the sequence $\{\operatorname{Supp}(H^i_{R_+}(M,N)_n)\}_{n\in\mathbb{Z}}$ is asymptotically stable for $n\to-\infty$. Thus there exists an integer $k$ such that $\operatorname{Supp}(H^i_{R_+}(M,N)_k)=\operatorname{Supp}(H^i_{R_+}(M,N)_n)$ for all $n\leq k$. Now, consider the

polynomial $Q(X)=\sum_{\mathfrak{p}\in \mathrm{Supp}(H^i_{R_+}(M,N)_k)} P_{H^i_{R_+}(M,N)_{\mathfrak{p}}}(X)e_d(\mathfrak{q}_0,R_0/\mathfrak{p})$. Then, by the associativity formula for the Samuel multiplicity, it is easily seen that $e_d(\mathfrak{q}_0,H^i_{R_+}(M,N)_n)=Q(n)$ for all $n\ll 0$. Thus we may assume that there are only finitely many $\mathfrak{p}\in \mathrm{Assh}(R_0)$ such that $\dim_{R_0}(H^i_{R_+}(M,N)_n)=2$. Then $\dim_{R_0}(H^i_{R_+}(M,N)_n)=1$, for all $n\ll 0$. By the graded Flat Base Change Theorem for generalized local cohomology we have

$$H^i_{R_+}(M,N)_n\otimes_{R_0}\widehat{R_0}\cong H^i_{(\widehat{R_0})_+}(\widehat{M},\widehat{N})_n,\ \text{for all}\, n\in\mathbb{Z},$$

where $\widehat{M}=M\otimes_{R_0}\widehat{R_0}$ and $\widehat{N}=N\otimes_{R_0}\widehat{R_0}$. As $\mathfrak{q}_0\widehat{R_0}$ is $\mathfrak{m}_0\widehat{R_0}$-primary, then, by Remark 4.2, we may (and do) assume that $(R_0,\mathfrak{m}_0)$ is complete. Now, from the short exact sequence $0\longrightarrow \Gamma_{\mathfrak{m}_0R}(N)\longrightarrow N\longrightarrow \bar{N}=N/\Gamma_{\mathfrak{m}_0R}(N)\longrightarrow 0$ we get the long exact sequence

$$H^i_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N))\longrightarrow H^i_{R_+}(M,N)\xrightarrow{\mu} H^i_{R_+}(M,\bar{N})\longrightarrow H^{i+1}_{R_+}(M,\Gamma_{\mathfrak{m}_0R}(N)). \tag{4.2}$$

According to Proposition 2.2(2) the outer modules in the exact sequence (4.2) are Artinian and so $\ker(\mu)$ and $\mathrm{coker}(\mu)$ are Artinian $R$-modules. Hence, by Definition 2.2(1), we deduce that $\ker(\mu)_n,\mathrm{coker}(\mu)_n$ are finite length $R_0$-modules, for all $n\in\mathbb{Z}$. Thus $e_d(\mathfrak{q}_0,\mathrm{coker}(\mu)_n)=e_d(\mathfrak{q}_0,\ker(\mu)_n)=0$, for all $n\ll 0$. Now, from the short exact sequence (4.2) we get the short exact sequences

$$0\longrightarrow \ker(\mu)\longrightarrow H^i_{R_+}(M,N)\longrightarrow \mathrm{im}(\mu)\longrightarrow 0$$

and

$$0\longrightarrow \mathrm{im}(\mu)\longrightarrow H^i_{R_+}(M,\bar{N})\longrightarrow \mathrm{coker}(\mu)\longrightarrow 0.$$

Using the fact that $e_d(\mathfrak{q}_0,-)$ is additive, for all $n\ll 0$, we have

$$e_d\big(\mathfrak{q}_0,H^i_{R_+}(M,N)_n\big)=e_d\big(\mathfrak{q}_0,\mathrm{im}(\mu)_n\big)=e_d\big(\mathfrak{q}_0,H^i_{R_+}(M,\bar{N})_n\big).$$

Therefore, without loss of generality (by replacing $N$ with $\bar{N}$) we may assume that $N$ is $\mathfrak{m}_0$-torsion-free, and so, by [4, Lemma 2.1.1(ii)], countable prime avoidance ([16, Theorem 2.1]) and Proposition 2.2(1), we deduce that there exists

$$x\in\mathfrak{m}_0\setminus\big(\bigcup\mathrm{Ass}(R_0)\big)\cup\big(\bigcup\mathrm{Ass}_{R_0}(N)\big)\cup\big(\bigcup_{i\in\mathbb{N}_0,\,n\in\mathbb{Z}}\mathrm{Ass}_{R_0}(H^i_{R_+}(M,\,N)_n)\setminus\{\mathfrak{m}_0\}\big)$$

which is a parameter element for $R_0$ and at the same time, $N$-regular and a filter regular element with respect to $H^i_{R_+}(M,N)_n$, for all $i\in\mathbb{N}_0$ and all $n\in\mathbb{Z}_0$. Set

$T_n := H^i_{R_+}(M,N)_n/\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)_n)$. As $x$ is $T_n$-regular and $\dim(T_n)=1$ for all $n \ll 0$, then it is concluded that, for all $n \ll 0$, we have

$$e_d(\mathfrak{q}_0, H^i_{R_+}(M,N)_n) = e_d(\mathfrak{q}_0, T_n/xT_n) = \ell_{R_0}(T_n/xT_n) = \ell_{R_0}(U_n) \tag{4.3}$$

where

$$U := \frac{H^i_{R_+}(M,N)}{\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N)) + xH^i_{R_+}(M,N)} \cong \frac{H^i_{R_+}(M,N)/\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N))}{x\big(H^i_{R_+}(M,N)/\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M,N))\big)}. \tag{4.4}$$

Now, the short exact sequence $0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$ gives the long exact sequence

$$\cdots \longrightarrow H^i_{R_+}(M,N) \xrightarrow{x} H^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/xN) \longrightarrow \cdots,$$

which in turn implies the monomorphism

$$0 \longrightarrow H^i_{R_+}(M,N)/xH^i_{R_+}(M,N) \longrightarrow H^i_{R_+}(M,N/xN). \tag{4.5}$$

By Proposition 2.3(2), $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N/xN)\big)$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less than $i$. As $\ell_{R_0}(U)_n$ is finite, for all $n \ll 0$, then, by linearity of length function and Proposition 2.3(2), for all $n \ll 0$, we have

$$\ell_{R_0}\Big(\frac{H^i_{R_+}(M,N)_n}{xH^i_{R_+}(M,N)_n}\Big) = \ell_{R_0}\Big(\frac{\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)_n) + xH^i_{R_+}(M,N)_n}{xH^i_{R_+}(M,N)_n}\Big) + \ell_{R_0}(U_n).$$

Thus, to complete the proof, it suffices to prove that $\ell_{R_0}(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N))_n$ is of reverse polynomial type of degree less than $i$. From the exact sequence (3.6), and Proposition 2.3(2), it is concluded that $\Gamma_{\mathfrak{m}_0 R}\big(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\big)$ is an Artinian graded $R$-module with Hilbert-Kirby polynomial of degree less that $i$. This means that, for all $n \ll 0$, we have

$$\begin{aligned}\ell_{R_0}\Big(\big(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\big)_n\Big) &= \ell_{R_0}\Big(\big(\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N))\big)_n\Big)\\ &= P_{\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M,N)/xH^i_{R_+}(M,N)\right)}(n).\end{aligned}$$

and we are done. □

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## References


[1] Brodmann, M., Fumasoli, S. and Tajarod, R., 2003. Local cohomology over homogeneous rings with one-dimensional local base ring. *Proc. Amer. Math. Soc.*, 131(10), pp.2977-2985. doi: 10.1090/S0002-9939-03-07009-6

[2] Brodmann, M. and Hellus, M., 2002. Cohomological patterns of coherent sheaves over projective schemes. *J. Pure Appl. Algebra*, 172(2-3), pp.165-182. doi: 10.1016/S0022-4049(01)00144-X

[3] Brodmann, M., Rohrer, F. and Sazeedeh, R., 2005. Multiplicities of graded components of local cohomology modules. *J. Pure Appl. Algebra*, 197(1-3), pp.249-278. doi: 10.1016/j.jpaa.2004.08.034

[4] Brodmann, M. and Sharp, R. Y., 2013. Local cohomology: an algebraic introduction with geometric applications (Vol. 136). Cambridge university press.

[5] Bruns, W. and Herzog, J., 1993. Cohen-macaulay rings (No. 39). Cambridge university press.

[6] Lech, C., 1957. On the associativity formula for multiplicities. *Ark. Mat.*, 3(4), pp.301-314. doi: 10.1007/BF02589424

[7] Cuong, N. T. and Van Hoang, N., 2005. Some finite properties of generalized local cohomology modules. *East West Math.*, 7(02), pp.107-115. doi: 10.645-1-10-20200302

[8] Cuong, N. T. and Van Hoang, N., 2008. On the vanishing and the finiteness of supports of generalized local cohomology modules. *Manuscripta Math.*, 126(1), pp.59-72. doi: 10.1007/s00229-007-0162-7

[9] Divaani-Aazar, K. and Hajikarimi, A., 2011. Generalized local cohomology modules and homological Gorenstein dimensions. *Comm. Alg.*, 39(6), pp.2051-2067. doi: 10.1080/00927870903295380

[10] Enochs, E. E. and Jenda, O. M., 2011. Relative homological algebra: Volume 1 (Vol. 30). Walter de Gruyter.

[11] Herzog, J., 1974. Komplexe, Auflösungen und Dualität in der lokalen Algebra. Habilitationsschrift, Universität Regensburg.

[12] Jahangiri, M., Shirmohammadi, N. and Tahamtan, S., 2015. Tameness and Artinianness of Graded Generalized Local Cohomology Modules. *Algebra Colloq.* Vol. 22, No. 01, pp. 131-146. doi: 10.1142/S1005386715000127

[13] D. Kirby Kirby, D., 1973. Artinian modules and Hilbert polynomials. *Q. J. Math.*, 24(1), pp.47-57. doi: 10.1093/qmath/24.1.47

[14] Shafiei, M., Khojali, A., Azari, A. and Zamani, N., 2021. Hilbert-Kirby Polynomials in Generalized Local Cohomology Modules. *Acta Math. Vietnam.*, 46, pp.747-759. doi: 10.1007/s40306-021-00440-3

[15] Rotthaus, C. and Şega, L., 2006. Open loci of graded modules. *Trans. Amer. Math. Soc.*, 358(11), pp.4959-4980. doi: 10.1090/s0002-9947-06-03876-1

[16] Sharp, R. Y. and Vámos, P., 1985. Baire's category theorem and prime avoidance in complete local rings. *Arch. Math.*, 44, pp.243-248. doi: 10.1007/BF01237858

[17] Suzuki, N., 1978. On the generalized local cohomology and its duality. *J. Math. Kyoto Univ.*, 18(1), pp.71-85. doi: 10.1215/kjm/1250522630

[18] Zamani, N., 2003. On the homogeneous pieces of graded generalized local cohomology modules. *Colloq. Math.* (Vol. 97, No. 2, pp. 181-188). doi:10.4064/cm97-2-5

[19] Zamani, N., 2006. On graded generalized local cohomology. *Arch. Math.*, 86(4), pp.321-330. doi: 10.1007/s00013-005-1524-6

[20] Zamani, N. and Khojali, A., 2015. Artinian graded generalized local cohomology. *J. Algebra Appl.*, 14(07), p.1550111. doi: 10.1142/S021949881550111X

**S. Karimbeyglu**
Department of Mathematics,
Faculty of sciences,
University of Mohaghegh Ardabili,
Ardabil, Iran
Email: saiedeh.Karimbaglou@gmail.com

**A. Khojali**
Department of Mathematics,
Faculty of sciences,
University of Mohaghegh Ardabili,
Ardabil, Iran
Email: khojali@uma.ac.ir

**N. Zamani**
Department of Mathematics,
Faculty of sciences,
University of Mohaghegh Ardabili,
Ardabil, Iran
Email: naserzak@yahoo.com

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