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QHC-SPACE CHARACTERIZATIONS

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Abstract. The aim of this article is to provide a detailed description of QHC-spaces. In addition we prove some results relating QHC-spaces with simple expansion topology and maximal QHC-spaces. In the last part, question posed by D. E. Cameron is almost resolved, that is, the necessary condition for QHC-space to be maximal has been stated.

1. Introduction

A topological space on a set X with topology τ shall be denoted (X, τ) and if τ and τ^* are two topologies on the same set, we say that τ^* is finer than of τ if $\tau \subseteq \tau^*$. A topological space (X, τ) with property R is maximal R if whenever τ^* is finer than τ, then (X, τ^*) does not have property R. If $A \subseteq X$ and $A \notin \tau$, then the topology with subbase $\tau \cup \{A\}$ is called the simple expansion of τ by A and is denoted $\tau(A)$.
For a topological space (X, τ) the subspace topology on a subset A of X is τ_A. We shall write $A^c, \mathrm{cl}(A), \mathrm{int}(A)$ and $\mathrm{bd}(A)$ for the complement, closure, interior and boundary respectively of a set A with respect to a topology τ. We shall subscript these sets if several different topologies are being discussed; for example, if τ and τ^* are topologies on X and $A \subseteq X$, then $\mathrm{cl}_\tau(A)$ and $\mathrm{cl}_{\tau^*}(A)$ are the closures of A with respect to τ and τ^* respectively.

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Definition 1.1. The topological space X is called a quasi H-closed space (briefly: QHC) if for each τ-open cover $\{U_i\}_{i\in I}$ of X there exists a finite subset I_0 of I such that $X = \bigcup_{i\in I_0} \mathrm{cl}(U_i)$. A subset A of the topological space X is said to be a QHC-subset if $\{U_i\}_{i\in I}$ is a family of τ-open sets of X that covers A, then there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{i\in I_0} \mathrm{cl}(U_i)$. A subset Y of the topological space X is a QHC-subspace if it is a QHC-space with respect to the subspace topology induced from X. A subset A of X is interiorly quasi H-closed (briefly: $IQHC$) if for each τ-open cover $\{U_i\}_{i\in I}$ of X there exists a finite subset I_0 of I such that $\mathrm{int}(A) \subseteq \bigcup_{i\in I_0} \mathrm{cl}(U_i)$.

It is clear if a subset A of a space (X,τ) is a QHC-subset, then it is $IQHC$, but not conversely (see [4] p. 315). In [4] it is observed that in a Hausdorff space an open set which is not closed cannot be $IQHC$, however in a T_1 space, open sets which are not closed may be $IQHC$.

It is easy to see that if A is a QHC-subspace of the topological space X, then it is a QHC-subset of X. It is not hard to find examples that show the converse of this fact is not true in general ([8] p. 161 and [6] Example 2.1).

From its definition it is obvious that every compact space is QHC, but not conversely (see [9] p. 127 and [1] p.119-120). It is well known that a regular QHC-space is compact. An alternate characterization is given by the following.

Proposition 1.2. *Let (X,τ) be a regular space and $A \subseteq X$. Then the following are equivalent:*

(1) A is a QHC-subspace.

(2) A is a QHC-subset.

(3) A is compact.

Proof. we prove only (2) $\Rightarrow$ (3), the argument to prove the remaining cases is analogous. Let $\mathcal{U}$ be an τ-open cover of A. By hypothesis, for each $a \in A$ and $U_a \in \mathcal{U}$ there exists a neighbourhood V_a of a such that $a \in V_a \subseteq \mathrm{cl}(V_a) \subseteq U_a$. Clearly, $A \subseteq \bigcup_{a\in A} V_a$ and so there is a finite subset F in A such that

$$A \subseteq \bigcup_{a\in F} V_a \subseteq \bigcup_{a\in F} \mathrm{cl}(V_a) \subseteq \bigcup_{a\in F} U_a.$$

□

It has been observed [2] that unlike compactness, a closed subset of a QHC topological space is not necessarily a QHC-subset. This difference in structure is the main reason for a difference in the properties of QHC-space and compact [3]. The following is an example of a Hausdorff QHC-space which is not compact and also its closed subset is not a QHC-subset.

Example 1.3. Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a family of pairwise distinct sets and $a \notin \bigcup_{n\in\mathbb{N}} X_n$. Define the following topology on $Y = \{a\} \cup \bigcup_{n\in\mathbb{N}} X_n$ by

$$\mathcal{O}Y = \{A : A \subseteq \bigcup_{n\in\mathbb{N}} X_n\} \cup \{U \subseteq Y : a \in U, \exists k \in \mathbb{N}, \bigcup_{n\geq k} X_n \subseteq U\}.$$

Let $X_n^* = X_n \cup \{\sigma_n\}$ be an one-point compactification of X_n for all $n \in \mathbb{N}$ and $\sigma_i \neq \sigma_j$ for all $i \neq j$. Now we define:

$$\mathcal{B} = \mathcal{O}Y \bigcup_{n\in\mathbb{N}} \mathcal{O}X_n^*.$$

It is clear that $\mathcal{B}$ is a base for a topology of the space X, which contains Y and X_n as subspaces. So since $D = \{\sigma_n : n \in \mathbb{N}\}$ has no any limit point in X, then X is not compact. To show that X is a QHC-space, suppose that $\mathcal{U}$ is an open cover of X. Since $a \in \bigcup\mathcal{U}$, then there exists $k \in \mathbb{N}$ such that $\bigcup_{n\geq k} X_n \subseteq U_0 \in \mathcal{U}$. For $n \leq k$, $\sigma_n \in \bigcup\mathcal{U}$ thus there are $U_n \in \mathcal{U}$ and finite subset $F_n \subseteq X_n$ such that $X_n \backslash F_n \subseteq U_n \in \mathcal{U}$. Consider $F = \bigcup_{n\leq k} F_n$. Now F is finite, therefore there are $U_k, \cdots, U_m \in \mathcal{U}$ such that $F \subseteq \bigcup_{n=k}^{m} U_m$. As a result $Y \subseteq \bigcup_{n=0}^{m} U_n$ and so

$$X = \mathrm{cl}(Y) = \mathrm{cl}(\bigcup_{n\geq k} X_n) \cup \bigcup_{n\leq k}(X_n \cup \{\sigma_n\}) \subseteq \bigcup_{n=0}^{m} \mathrm{cl}(U_n).$$

Therefore X is a QHC-space. To show that a closed subset is not necessarily QHC, it suffices to consider $D = \{\sigma_1, \cdots, \sigma_m, \cdots\}$ as a closed subset of X. Since D is a discrete subset, then it can not be a QHC-subset.

In the following, we express the difference between compact spaces and QHC-spaces. We know that for any continuous function from a compact space to a Haussdorff space,

the inverse image of any compact subset is compact. But the following example shows that if X is a QHC-space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous function, then the inverse image of any QHC-subspace of Y may not necessarily be a QHC-space.

Example 1.4. Let X be the space defined in the Example 1.3 with the condition that the discrete space X_n is countable for each $n \in \mathbb{N}$, i.e, $X_n = \{x_{n1}, x_{n2}, \cdots, x_{mn}, \cdots\}$. Now we choose the real sequence $\{r_{1j}\}_{j\in\mathbb{N}}$ such that it is strictly decreasing and converges to one. In addition, we define a real sequence $\{r_{ij}\}_{j\in\mathbb{N}}$ for each $i > 1$ such that it is strictly descending and converges to $\frac{1}{i}$ and all its elements are less than $\frac{1}{i-1}$. Define the function $f : X \to \mathbb{R}$ as follows:

$$\forall n \in \mathbb{N} \quad f(x_{nj}) = r_{nj}, \quad f(\sigma_n) = \frac{1}{n}, \quad f(a) = 0.$$

It is obvious that f is a continuous function and

$$f^{-1}(\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}) = \{a, \sigma_1, \sigma_2, \cdots, \sigma_n, \cdots\}.$$

It is known that $\{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$ is compact and QHC-space, but $\{a, \sigma_1, \sigma_2, \cdots, \sigma_n, \cdots\}$ is not the same.

2. QHC-space

In this section, We give some characterizations of QHC-spaces. The following characterization of QHC-spaces is most useful in our discussion.

Theorem 2.1. *For a topological space (X, τ), the following implications hold.*
(1)[4]*. X is QHC if and only if every proper regular closed subset of X is QHC (A closed set A is regular closed if $A = \mathrm{cl}(\mathrm{int}(A))$ or equivalently A is the closure of an open set).*
(2)[4]*. X is QHC if and only if every closed subset of X is $IQHC$.*
(3)[5]*. Let X be Hausdorff and a set $A \subseteq X$ be QHC-subset. Then A is closed.*

It is not difficult to observe that the continuous image of a QHC-space (QHC-subset) is a QHC-space (resp. QHC-subset). Thus, by this fact and Theorem 2.1, we obtain the following.

Corollary 2.2. *The continuous image of any QHC-subset to the Hausdorff space, is closed.*

Theorem 2.3. *Let (X,τ) be a topological space. Then the following hold:*
(1) QHC is preserved by finite unions.
(2) The closure of a QHC-subspace is a QHC-subspace.
(3) If A is a closed subset of a QHC-space X and if the boundary of A is a QHC-subspace, then so is A.
(4) A QHC-space is compact if and only if each closed subset is QHC.
(5) The QHC property is contractive (A topological property is contractive if τ^* has property R whenever τ has property R and τ is finer than τ^*).

Proof. (1) Let $\{U_i\}_{i\in I}$ be an arbitrary τ-open cover of $A\cup B$. By assumption, there exist finite subsets $I_1, I_2 \subseteq I$ such that $A \subseteq \bigcup_{i\in I_1} \mathrm{cl}(U_i)$ and $B \subseteq \bigcup_{i\in I_2} \mathrm{cl}(U_i)$. Since $I_1 \cup I_2$ is a finite subset of I, then $A\cup B \subseteq (\bigcup_{i\in I_1} \mathrm{cl}(U_i)) \cup (\bigcup_{i\in I_2} \mathrm{cl}(U_i)) = \bigcup_{i\in I_1\cup I_2} \mathrm{cl}(U_i)$.
(2) Let $A \subseteq X$ be any QHC-subspace. For the purpose of showing that $\mathrm{cl}(A)$ also is a QHC-subspace, we take any τ_A-open cover $\{U_i \cap \mathrm{cl}_\tau(A)\}_{i\in I}$ of $\mathrm{cl}_\tau(A)$. Since A is a QHC-subspace, then there exists a finite subset $I_0 \subseteq I$ such that $A \subseteq \bigcup_{i\in I_0} \mathrm{cl}_{\tau_A}(U_i \cap A)$. Hence $A \subseteq \bigcup_{i\in I_0} \mathrm{cl}_\tau(U_i \cap A)$. Moreover this implies $\mathrm{cl}_\tau(A) \subseteq \bigcup_{i\in I_0} \mathrm{cl}_\tau(U_i \cap A)$, then $\mathrm{cl}_\tau(A) \subseteq \bigcup_{i\in I_0} \mathrm{cl}_\tau(U_i \cap \mathrm{cl}_\tau(A))$. Therefore $\mathrm{cl}_\tau(A) \subseteq \bigcup_{i\in I_0} \mathrm{cl}_{\tau_{\mathrm{cl}_\tau(A)}}(U_i \cap \mathrm{cl}_\tau(A))$, which means $\mathrm{cl}(A)$ also is a QHC-subspace.
(3) Let $\{V_i\}_{i\in I}$ be an arbitrary τ_A-open cover of A. Expand every V_i to U_i open in X such that $V_i = U_i \cap A$. Note that A is closed in X so A^c is open and also $\mathrm{bd}(A) \subseteq A$. Now make two new covers $\{U_i \cap \partial(A)\}_{i\in I}$ of $\mathrm{bd}(A)$ and $\{U_i\}_{i\in I} \cup \{A^c\}$ of X, and both of these spaces are by assumption QHC. So finite subsets of each of them are dense in their respective space, so there are finite subsets $I_0, I_1 \subseteq I$ such that $X = \bigcup_{i\in I_0} \mathrm{cl}_\tau(\{U_i\} \cup A^c)$ and $\mathrm{bd}(A) \subseteq \bigcup_{i\in I_1} \mathrm{cl}_\tau(U_i \cap \mathrm{bd}(A))$, which can easily be checked $A \subseteq \bigcup_{i\in I_0\cup I_1} \mathrm{cl}_\tau(U_i)$.
(4) Necessity. suppose that X is compact. Each closed subset of X is also compact since any closed subset of a compact set is compact and hence, each closed subset is QHC.
Sufficiency. suppose that each closed subset of X is a QHC-subset. In particular, X is QHC as a closed subset of itself. Let U_i be any open cover of X. Since X is closed, it has a finite subcover of closed sets V_i. However, just use V_i^c to arrive at a finite subcover.
(5) Let $\tau \subseteq \tau^*$ be topologies on X and (X,τ^*) be a QHC-space. If $\mathcal{U} = \{V_i\}_{i\in I}$ is a

τ-open cover of X, then it is also a τ^*-open cover of X. By assumption, there exists a finite subset $I_0 \subseteq I$ such that $X = \bigcup_{i\in I_0} \mathrm{cl}_{\tau^*}(V_i)$. Since $\mathrm{cl}_{\tau^*}(V_{i_j}) \subseteq \mathrm{cl}_{\tau}(V_{i_j})$ for all $i \in I_0$, so $X = \bigcup_{i\in I_0} \mathrm{cl}_{\tau^*}(V_i) \subseteq \bigcup_{i\in I_0} \mathrm{cl}_{\tau}(V_i)$. □

Similarly we can prove the following:

Proposition 2.4. *Let $\tau \subseteq \tau^*$ be topologies on X and $A \subseteq X$. Then the following statements hold:*
(1) If A is a $IQHC$-subset in (X, τ^*), it is the same in (X, τ).
(2) If A is a QHC- subset in (X, τ^*), it is the same in (X, τ).
(3) If A is a QHC-space in (X, τ^*), it is the same in (X, τ).

Proof. These are immediate consequences of Theorem 2.3(4). □

Remark 2.5. Note that proposition 2.3 (2) is not valid for $IQHC$-subsets. Let $X = \mathbb{R}$ with the usual topology, and let $A = \mathbb{Q}$. Trivially, A is a $IQHC$-subset, but $\mathrm{cl}(A) = \mathbb{R}$ is not the same.

Theorem 2.6. [5] *Let (X, τ) be a space and $A, B \subseteq X$. If A is a QHC-subset and B is a closed and open subset in (X, τ), then $A \cap B$ is a QHC-subset in (X, τ).*

Corollary 2.7. *For every QHC-space (X, τ), every closed and open subset of X is a QHC-subset in (X, τ).*

Definition 2.8. A subset A of (X, τ) is said to be α-open if $A \subseteq \mathrm{int}\big(\mathrm{cl}(\mathrm{int}(A))\big)$. The family of all α-open subset of (X, τ) is denoted by τ^α. For every space the family τ^α forms a topology on X.

Theorem 2.9. [5]. *Let (X, τ) be a topological space and let α-open sets A, B be such that $A \cap B = \emptyset$. Then $A \cup B$ is a QHC-subset if and only if A and B are both QHC-subsets.*

Corollary 2.10. *Let A be a closed and open subset in (X, τ). Then the space (X, τ) is QHC if and only if A and A^c are QHC-subspaces.*

Lemma 2.11. *Let (X, τ) be a topological space and $A, B \subseteq X$. If $\mathrm{bd}(A) \cap \mathrm{bd}(B) = \emptyset$, then $\mathrm{int}(A \cup B) = \mathrm{int}(A) \cup \mathrm{int}(B)$*

Proof. Necessity. It is obvious.
Sufficiency. We claim if $x \in \text{int}(A \cup B)$ and $x \notin \text{int}(A) \cup \text{int}(B)$, then $x \in \text{int}(A) \cap \text{int}(B)$. Let $x \in \text{int}(A)$. Since $x \notin \text{int}(A)$, it is enough to show $x \in \text{cl}(A)$. Let U be an open neighborhood of x, so $V = U \cap \text{int}(A \cup B)$ is an open neighborhood of x and $V \subseteq A \cup B$. However $V \subseteq B$ and $x \notin \text{int}(B)$, then $V \cap A \neq \emptyset$, so $U \cap A \neq \emptyset$. □

Theorem 2.12. *Let (X, τ) be a topological space and $A, B \subseteq X$. If A and B are both $IQHC$- subset such that $\text{bd}(A) \cap \text{bd}(B) = \emptyset$, then $A \cup B$ is a $IQHC$- subset.*

Proof. Let $\{V_i\}_{i \in I}$ be an arbitrary τ-open cover of $A \cup B$. Since $\{V_i\}_{i \in I}$ a τ-open cover of A and also B, so there exist finite subsets $I_0, I_1 \subseteq I$ such that $\text{int}(A) \subseteq \bigcup_{I_0 \in I} \text{cl}(V_i)$ and $\text{int}(B) \subseteq \bigcup_{I_1 \in I} \text{cl}(V_i)$. Therefore $\text{bd}(A) \cup \text{bd}(B) \subseteq \bigcup_{I_0 \cup I_1 \in I} \text{cl}(V_i)$, by Lemma 2.11, $\text{int}(A \cup B) \subseteq \bigcup_{I_1 \in I} \text{cl}(V_i)$. □

Definition 2.13. A family $\mathcal{A}$ of subsets of the topological space X has the finite interior intersection property, if the intersection of interiors of sets in each finite subfamily of $\mathcal{A}$ is non-empty. In other words if $A_1, \cdots, A_n \in \mathcal{A}$, then $\bigcap_{i=1}^{n} \text{int}(A_i) \neq \emptyset$.

Proposition 2.14. *The topological space X is QHC if and only if for every family of closed subsets $\{F_i\}_{i \in I}$ in X satisfying the finite interior intersection property, $\bigcap_{i \in I} F_i \neq \emptyset$.*

Proof. Necessity. Let $\{F_i\}_{i \in I}$ be a family of closed subsets of X such that $\bigcap_{i \in I} F_i = \emptyset$. Since $X = \bigcup_{i \in I} F_i^c$, $\{F_i^c\}_{i \in I}$ is an open cover of X. Thus there exists a finite subset I_0 of I such that $X = \bigcup_{i \in I_0} \text{cl}(F_i^c) = \bigcup_{i \in I_0} (\text{int}(F_i))^c$. Therefore $\bigcap_{i \in I_0} \text{int}(F_i) = \emptyset$.
Sufficiently. If $\{F_i\}_{i \in I}$ is the family of closed subset in X satisfying the interior finite intersection property, then for each finite subsets I_0 of I, $\bigcap_{i \in I_0} \text{int}(F_i) = \emptyset$. Let $\mathcal{C}$ be an open cover of X. Then $X = \bigcup_{C \in \mathcal{C}} C$ implies $\emptyset = \bigcap_{C \in \mathcal{C}} C^c$. Since $\mathcal{F} = \{C^c | C \in \mathcal{C}\}$ is a family of closed subset in X, so there is a finite subcollection $\{C_1^c, \cdots, C_n^c\}$ of $\mathcal{F}$, such that $\emptyset = \bigcap_{i=1}^{n} \text{int}_\tau(C_i^c) = \bigcap_{i=1}^{n} (\text{cl}_\tau(C_i))^c$. It follows that $\bigcup_{i=1}^{n} \text{cl}(C_i) = X$ and (X, τ) is a QHC-space. □

Using the same basic method of proof of proposition 2.14 we can also prove the following:

Proposition 2.15. *Let X be a the topological space and $Y \subseteq X$. Then Y is a QHC-subset if and only if for every family of closed subsets $\{F_i\}_{i\in I}$ in X included in Y satisfying the finite interior intersection property, $\bigcap_{i\in I} F_i \neq \emptyset$.*

Proof. It is obvious. □

Proposition 2.16. *If X is a QHC-space and T_1, then the following hold:*
(1) Every infinite subset with finite boundary has a limit point.
(2) Every infinite open subset X has a limit point.

Proof. (1) Let $A \subseteq X$ be an infinite subset with finite boundary that has no any limit points. So A is closed and discrete. Hence for all $a \in A$ there is an open neighborhood U_a of a such that $U_a \cap A = \{a\}$. Furthermore, $\{U_a : a \in A\} \cup \{A^c\}$ is an open cover of X. So there are $a_1, \cdots, a_n \in A$ such that $X = \mathrm{cl}(A^c) \cup \mathrm{cl}(\bigcup_{i=1}^{n} U_{a_i})$. So

$$A = (A \cap \mathrm{cl}(A^c)) \cup \mathrm{cl}(\bigcup_{i=1}^{n}(A \cap U_{a_i}) = \mathrm{bd}(A) \cup \mathrm{cl}(\bigcup_{i=1}^{n}(A \cap U_{a_i}) = \mathrm{bd}(A) \cup \{a_i : i = 1, \cdots, n\},$$

and it contradicts the fact that A is infinite.
(2) Assume that A has no any limit points. Therefore A is a closed and open set and consequently $\mathrm{bd}(A)$ is finite. According to (1), the set A has a limit point, which is a contradiction. □

The following fact of topological space X, which is used in proof of the next theorem, is well-known and so the proof is omitted.

Lemma 2.17. *Let D be a dense subset of X and let U be an open set of X. Then $\mathrm{cl}(D \cap U) = \mathrm{cl}(U)$.*

Therefore, if D_1 and D_2 are two open dense subsets of X, then by the above Lemma, we can write $\mathrm{cl}(D_1 \cap D_2) = \mathrm{cl}(D_1) = X$.

Theorem 2.18. *Let $\mathcal{S}$ be a subbase for a topological space X. Then X is a QHC-space if and only if $\mathcal{C}$ is a cover of X such that $\mathcal{C} \subseteq \mathcal{S}$ then $X \subseteq \bigcup_{i=1}^{n} \mathrm{cl}(C_i)$, for some $C_1, \cdots, C_n \in \mathcal{C}$.*

Proof. Necessity. It is clear.

Sufficiently. Assume that X is not a QHC-space. We choose a maximal open covering $\mathcal{C}$ which dose not have a finite cover whose closures cover X. Note that $\mathcal{C} \cap \mathcal{S}$ can not cover X, then there exists $x_0 \in X \setminus \cup(\mathcal{C} \cap \mathcal{S})$. Since $\mathcal{C}$ is an open cover of X, there exists an open set $U_0 \in \mathcal{C}$ such that $x_0 \in U_0$. By assumption, there are finite subcollection of subbases elements $S_1, \cdots, S_n \in \mathcal{S}$ such that $x_0 \in \bigcap_{i=1}^{n} S_i \subseteq U_0$. Now since $\mathcal{C}$ is the maximal element, we must have an open cover $\mathcal{C} \cup \{S_i\}$ for each $1 \leq i \leq n$ that has a finite cover whose closure covers X, i.e there is $\mathcal{F}_i \subseteq \mathcal{C}$ for all $1 \leq i \leq n$ such that $X = \mathrm{cl}(S_i) \cup \bigcup_{U \in \mathcal{F}_i} \mathrm{cl}(U)$. Consider $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$, $W = \bigcup_{U \in \mathcal{F}} \mathrm{cl}(U)$ and $D_i = S_i \cup W$ for all $1 \leq i \leq n$. It is obvious that D_i is an open dense subset in X for all $1 \leq i \leq n$, and therefore $D = \cap_{i=1}^{n} D_i$ is also dense in X. Hence

$$\begin{aligned} X = \mathrm{cl}(D) = \mathrm{cl}(\bigcap_{i=1}^{n} D_i) &= \mathrm{cl}(\bigcap_{i=1}^{n} S_i \cup W) \\ &= \mathrm{cl}((\bigcap_{i=1}^{n} S_i) \cup W) \\ &= \mathrm{cl}(\bigcap_{i=1}^{n} S_i) \cup \mathrm{cl}(W) \\ &\subseteq \mathrm{cl}(U_0) \cup \mathrm{cl}(W). \end{aligned}$$

Furthermore, $\mathcal{F} \cup \{U_0\}$ is a finite cover of $\mathcal{C}$ whose closures cover X, which is the desired contradiction. □

3. QHC-space and QHC-subset

It is well known that a subset A of a space (X, τ) can be QHC-subset without being QHC-subspace. In this section, we present some sufficient conditions for the converse to hold.

Definition 3.1. A subset A of (X, τ) is said to be pre-open if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. The family of all pre-open subset of (X, τ) is denoted by $PO(X, \tau)$.

Proposition 3.2. [7] *A set $A \in PO(X, \tau)$ is a QHC-subspace of (X, τ) if and only if it is a QHC-subset.*

Proposition 3.3. *Let (X, τ) be a topological space. Then the following hold:*
(1) Every dense QHC-subset is a QHC-space.
(2) Supposed that Y is a closed QHC-subset in X and $A \subseteq Y$ is a closed subset in X. If $\text{int}(A) \neq \emptyset$ in Y implies that $\text{int}_\tau(A) \neq \emptyset$, then Y is a QHC-space.

Proof. (1) Let A be a dense and QHC-subset in X and $A = \bigcup_{U \in \mathcal{U}} U \cap A$, where $\mathcal{U}$ is a τ-open in X. Then $A \subseteq \bigcup_{U \in \mathcal{U}} U$ and there are $U_1, \cdots, U_n \in \mathcal{U}$ such that $A \subseteq \bigcup_{i=1}^{n} \text{cl}(U_i) \subseteq \bigcup_{i=1}^{n} \text{cl}(U_i \cap A)$. Hence

$$A = \bigcup_{i=1}^{n} \text{cl}(U_i \cap A) \cap A = \bigcup_{i=1}^{n} \text{cl}_{\tau_A}(U_i \cap A).$$

(2) Follows from Proposition 2.15. □

It is clear that in the proposition 3.3 (2), if X is Hausdorff, the assumption that Y is closed is redundant. It can also be said that the assumption that Y is closed is not a necessary condition for the proposition to be true. For example, if X has a finite complement topology, then proper infinite subsets of X are not closed and the condition given in (2) holds. Moreover, every subset of X is compact and therefore a QHC-space. Proposition 3.3 raised the question of whether the converse is true if X is Hausdorff. It can be easily seen that the answer to this question is negative. For example, let X be a Hausdorff space with no isolated points. Let $Y \subseteq X$ be finite. Hence Y is a QHC-space and for every $\emptyset \neq A \subseteq Y$, $\text{int}(A) = A \neq \emptyset$ in Y, while $\text{int}_\tau(A) = \emptyset$.
In the following, we will expand Theorem 2.6 and Corollary 2.7 using the next propositions and lemmas.

Proposition 3.4. *Let (X, τ) be a topological space and $A \subseteq X$. Then the following statements hold:*
(1) If A and A^c are QHC-subsets, then X is a QHC-space.
(2) Let $A \subseteq Y \subseteq X$. If Y is a QHC-subset in X and A is a closed and open subset in X, then A is a QHC-subspace in X.
(3) If A is a closed and open subset in QHC-space X, then A is a QHC-space.
(4) Every open QHC-subset A in (X, τ) is a QHC-space.

(5) If A is closed and open subset in X, then the following hold:

(a) X is a QHC-space.

(b) Both A and A^c are QHC-spaces.

(c) Both A and A^c are QHC-subsets.

Proof. We only prove (2), the rest of the proofs are straightforward. Let $\{A \cap U_i\}_{i\in I}$ be an arbitrary τ_A-open cover of A in Y, where U_i is an τ-open subset in X. Since $Y \subseteq (\bigcup_{i\in I} U_i) \cup A^c$ and A^c is open in X, then there are $i_1, \cdots, i_n \in I$ such that $Y \subseteq (\bigcup_{j=1}^{n} \mathrm{cl}_\tau(U_{i_j})) \cup \mathrm{cl}_\tau(A^c)$. Therefore

$$A = (\bigcup_{j=1}^{n} A \cap \mathrm{cl}_\tau(U_{i_j})) \cup (A \cap \mathrm{cl}_\tau(A^c)) = \bigcup_{j=1}^{n} A \cap \mathrm{cl}_\tau(U_{i_j}) = \bigcup_{j=1}^{n} \mathrm{cl}_\tau(A \cap U_{i_j}).$$

□

Lemma 3.5. *If A, B are distinct α-open sets in (X, τ), then A, B are open subsets in $A \cup B$.*

Proof. It is enough to show that A is an open subset in $A \cup B$. Let $a \in A$, then there is an open subset U such that $a \in U \subseteq \mathrm{cl}(\mathrm{int}(A))$. Hence

$$\mathrm{int}(A) \cap \mathrm{int}(B) = \emptyset \Rightarrow \mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(B) = \emptyset \Rightarrow U \cap \mathrm{int}(B) = \emptyset.$$

Therefore

$$U \cap \mathrm{cl}(\mathrm{int}(B)) = \emptyset \Rightarrow U \cap B = \emptyset \Rightarrow U \cap (A \cup B) = U \cap A \subseteq A.$$

As a result $a \in \mathrm{int}(A)$ in $A \cup B$. □

In order to prove the preceding proposition, we need to prove the following auxiliary results:

Lemma 3.6. *If U is an open subset in (X, τ) and $A \in \tau^\alpha$, then $A \cap \mathrm{cl}_\tau(U) = \mathrm{cl}_{\tau_A}(A \cap U)$.*

Proof. It is enough to prove that $A \cap \mathrm{cl}_\tau(U) \subseteq \mathrm{cl}_{\tau_A}(A \cap U)$. Consider a point $a \in A \setminus \mathrm{cl}_{\tau_A}(A \cap U)$. We must show that $a \notin \mathrm{cl}_\tau(U)$. Since $a \in A \subseteq \mathrm{int}(\mathrm{cl}_\tau(\mathrm{int}(A)))$, then there is an open neighborhood W such that $(W \cap A) \cap (A \cap U) = \emptyset$ and $W \subseteq \mathrm{cl}_\tau(\mathrm{int}(A))$. Hence $W \cap (A \cap U) = \emptyset$ and $W \subseteq \mathrm{cl}_\tau(\mathrm{int}(A))$. Therefore $W \cap U \cap \mathrm{cl}_\tau(\mathrm{int}(A)) = \emptyset$ and $W \subseteq \mathrm{cl}_\tau(\mathrm{int}(A))$. This show that $W \cap U = \emptyset$ and $a \notin \mathrm{cl}_\tau(U)$. □

We are now ready to improve 2.6 as follows:

Proposition 3.7. *Let A, B be α-open sets in (X, τ). Then the following statements are equivalent:*

(1) $A \cup B$ is a QHC-space.

(2) A, B are QHC-spaces.

(3) A, B are QHC-subsets.

(4) $A \cup B$ is a QHC-subset.

Proof. These follow from Proposition 3.4 (5), Lemma 3.5 and 3.6. □

4. Specifications of QHC-spaces with simple expansion topology

In this section we shall extend the concept of QHC-spaces with simple expansion topology. We shall show that in maximal QHC-spaces, QHC-subsets by compact boundary are closed subsets.

Theorem 4.1. *Let (X, τ) be a QHC-space and $A \subseteq X$. If the boundary of A is compact and A is QHC in (X, τ), then $(X, \tau(A^c))$ is QHC.*

Proof. Let A be QHC in (X, τ). Now, if $\mathcal{C} = \{U_i \cup (O_i \cap A^c) \mid i \in I, U_i, O_i \in \tau\}$ is an open cover of $(X, \tau(A^c))$, then $\{U_i\}_{i\in I}$ is an open cover of A with respect to τ, so by assumption there is a finite set $J \subseteq I$, so that $A \subseteq \bigcup_{i\in J} \mathrm{cl}_\tau(U_i)$. Therefore $\mathrm{cl}(A) \subseteq \bigcup_{i\in J} \mathrm{cl}_\tau(U_i) \quad (*)$. Of course the collection $\{U_i | i \in I\} \cup \{O_i | i \in I\}$ is open cover of (X, τ), so here we also have a finite $K, L \subseteq I$, that $X \subseteq (\bigcup_{i\in K} \mathrm{cl}_\tau(U_i)) \cup (\bigcup_{i\in L} \mathrm{cl}_\tau(O_i)) \quad (**)$. We show that $X \subseteq \bigcup_{i\in J\cup K\cup L} \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$. For all $x \in X$, we consider two items:

a: $x \in \mathrm{cl}(A)$. According to (*), there is $i \in J$, that $x \in \mathrm{cl}_\tau(U_i)$. Since for all open neighborhood $V = U \cup (O \cap A^c)$ of x in $\tau(A^c)$, $x \in U \in \tau$, then $x \in \mathrm{cl}_\tau(U_i)$ implies $\emptyset \neq U \cap U_i \subseteq V \cap U_i \subseteq V \cap (U_i \cup (O_i \cap A^c))$. Therefore $x \in \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$ that $i \in J$, so $x \in \bigcup_{i\in J\cup K\cup L} \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$.

b: $x \in \mathrm{cl}(A^c) = (\mathrm{int}(A))^c$. According to (**), there is $i \in L$ or $i \in K$, that $x \in \mathrm{cl}_\tau(U_i)$ or $x \in \mathrm{cl}_\tau(O_i)$.

1: For some $i \in K$, $x \in \mathrm{cl}_\tau(U_i)$. For all open neighborhood $V = U \cup (O \cap A^c)$ of x in $\tau(A^c)$, $x \in U$ or $x \in O \cap A^c$.

If $x \in U \in \tau$, then $\emptyset \neq U \cap U_i \subseteq V \cap U_i \subseteq V \cap (U_i \cup (O_i \cap A^c))$, therefore $x \in \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$. Since $i \in K$, $x \in \bigcup_{i \in J \cup K \cup L} \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$.

But if $x \in O \cap A^c$, by assumption $x \in \mathrm{int}(A^c)$ and since $x \in O = \mathrm{int}(O)$, so $x \in \mathrm{int}(O) \cap \mathrm{int}(A) = \mathrm{int}(O \cap A^c)$ and

$$\emptyset \neq U_i \cap (O \cap A^c) \subseteq U_i \cap V \subseteq V \cap (U_i \cup (O_i \cap A^c))$$

Then $x \in \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$ and since $i \in K$, $x \in \bigcup_{i \in J \cup K \cup L} \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$.

2: For some $i \in L$, $x \in \mathrm{cl}_\tau(O_i)$. Note that the open sets $O \cap A^c$ is a neighborhood basis for each point x in $\tau(A^c)$ consisting open set in $\tau(A^c)$, so there exists an open neighborhood $V = O \cap A^c$ for all elements of this basic, that $x \in \mathrm{int}(A^c), x \in O = \mathrm{int}(O)$, then $x \in \mathrm{int}(A^c) \cap \mathrm{int}(O) = (O \cap \mathrm{int}(A^c))$. Hence

$$\emptyset \neq O_i \cap (O \cap A^c) = (O_i \cap A^c) \cap (O \cap A^c) = (O_i \cap A^c) \cap V \subseteq (U_i \cup ((O_i \cap A^c) \cap)) \cap V$$

Then $x \in \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c))$ and since $i \in L$,

$$x \in \bigcup_{i \in J \cup K \cup L} \mathrm{cl}_{\tau(A^c)}(U_i \cup (O_i \cap A^c)).$$

□

Theorem 4.2. *Let (X, τ) be a QHC-space and $A \subseteq X$. If $(X, \tau(A^c))$ is QHC, then A is $IQHC$ in (X, τ)*

Proof. Let $(X, \tau(A^c))$ be a QHC-space. Since $A^c \in \tau(A^c)$, so A is closed in $(X, \tau(A^c))$ and by Theorem 2.1 is $IQHC$ in $(X, \tau(A^c))$. We know that $\tau \subseteq \tau(A^c)$, so by Proposition 4.2 implies that A is $IQHC$ in (X, τ). □

We end our study of QHC-spaces with simple expansion topology with the following theorems:

Theorem 4.3. *The topological QHC-space (X, τ) is maximal QHC if and only if for $A \notin \tau$, $(X, \tau(A))$ is not QHC.*

Proof. Necessity. It is obvious.

Sufficiency. If (X, τ) is not maximal QHC, then there exists a topology $\tau \subsetneq \tau^*$ on X so that (X, τ^*) is QHC. Consider $A \in \tau^*$, then $\tau \subseteq \tau(A) \subseteq \tau^*$. By Proposition 2.4, $\tau(A)$ is a QHC-space, which is a contradiction. □

Theorem 4.4. *Let (X,τ) be a maximal QHC-space. Then QHC-subsets by compact boundary are closed subsets.*

Proof. Let (X,τ) be QHC and $A \subseteq X$ is a $IQHC$- subset of X with compact boundary which is not closed, so $A^c \notin \tau$ and $\tau \subsetneq \tau(A^c)$. By Theorem 4.1, $(X,\tau(A^c))$ is QHC which is a contradiction since (X,τ) is maximal QHC. □

Theorem 4.5. *Let (X,τ) be a QHC-space. If every $IQHC$-subset is closed, then (X,τ) is a maximal QHC-space.*

Proof. Assume that (X,τ) is not maximal QHC, then there exists a topology $\tau \subsetneq \tau^*$ on X so that (X,τ^*) is QHC. Choose $A \in \tau^* \setminus \tau$, so A^c is a closed subset in τ^*, but not in τ. Now suppose $\{V_i\}_{i\in I}$ is an open cover for A in τ (i.e $A \subseteq \bigcup_{i\in I} V_i$ that $V_i \in \tau$). Since $\tau \subsetneq \tau^*$, so $\{V_i\}_{i\in I}$ is an open cover for A in τ^*. A^c is closed subset in τ^*, then by Theorem 2.1, A^c is a $IQHC$-subset in τ^*. Hence there are $V_1,\ldots,V_n$ so that $\mathrm{int}_{\tau^*}(A^c) \subseteq \bigcup_{i\in I}^{n} \mathrm{cl}_{\tau^*}(V_i)$. Sine $\tau \subsetneq \tau^*$, then $\mathrm{cl}_{\tau^*}(V_i) \subseteq \mathrm{cl}_{\tau}(V_i)$ and $\mathrm{int}_{\tau}(A^c) \subseteq \mathrm{int}_{\tau^*}(A^c)$. Therefore

$$\mathrm{int}_{\tau}(A^c) \subseteq \mathrm{int}_{\tau^*}(A^c) \subseteq \bigcup_{i\in I}^{n} \mathrm{cl}_{\tau^*}(A_i) \subseteq \bigcup_{i\in I}^{n} \mathrm{cl}_{\tau}(V_i)$$

In other words A^c is $IQHC$ in τ but not closed. □

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