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ON CERTAIN CLASSES OF GROUP DERIVATIONS

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Abstract. Let Γ be a group and $\mathcal{X}$ denote a class of (left) Γ-modules which contains all projectives. Assume that the trivial Γ-module $\mathbb{Z}$ admits an $\mathcal{X}$-precover. In this paper, we introduce and study a class of the derivations from a group Γ to a Γ-module A that arises from $\mathcal{X}$-precovers of the Γ-module $\mathbb{Z}$. These derivations, that will be called $\mathcal{X}$-derivations, form a subgroup of the group of derivations $\mathrm{Der}(\Gamma, A)$. We study this group explicitly when $\mathcal{X}$ is the class of Gorenstein projective Γ-modules, where $\mathcal{X}$-derivations are called Gorenstein derivations. Corresponding results will be discussed when $\mathcal{X}$ is the class of flat Γ-modules.

1. Introduction

In many algebraic structures such as groups, associative algebras and Lie algebras, the notion of a derivation plays an important role, especially in the study of extensions and cohomology [6], [17], [22]. The abelian group $\mathrm{Der}(\Gamma, A)$ of derivations (sometimes called crossed homomorphisms) from a group Γ to a Γ-module A appears in the first group cohomology $H^1(\Gamma, A)$. In fact, $H^1(\Gamma, A)$ is the group of derivations modulo principal derivations, denoted $\mathrm{PDer}(\Gamma, A)$. Also we know, from the classical homological algebra, that $H^1(\Gamma, A)$ is isomorphic to the group $e(\Gamma, A)$, the Yoneda Ext of the pair $(\mathbb{Z}, A)$, that

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is, the group of equivalence classes of short exact sequences $0 \to A \to E \to \mathbb{Z} \to 0$. So

$$e(\Gamma, A) \cong \mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A) \cong H^1(\Gamma, A) \quad (\dagger)$$

Let $\mathcal{X}$ be a class of (left) Γ-modules which contains all projective Γ-modules. The extension $0 \to A \to E \to \mathbb{Z} \to 0$ is called $\mathcal{X}$-*proper* if it is $\mathrm{Hom}_\Gamma(\mathcal{X}, \ )$-exact. Let $e_\mathcal{X}(\Gamma, A)$ be the subset of $e(\Gamma, A)$ consisting of all equivalence classes of $\mathcal{X}$-proper extensions of $\mathbb{Z}$ by A. It is proved in [1] that $e_\mathcal{X}(\Gamma, A) \cong \mathrm{Ext}^1_\mathcal{X}(\mathbb{Z}, A)$ is a subgroup of $\mathrm{Ext}^1_{\mathbb{Z}\Gamma}(\mathbb{Z}, A) \cong H^1(\Gamma, A)$. Here $\mathrm{Ext}^1_\mathcal{X}(\mathbb{Z}, A)$ is the 1st right derived functor of $\mathrm{Hom}_{\mathbb{Z}\Gamma}(\ , A)$ with respect to $\mathcal{X}$. It follows from the isomorphism $(\dagger)$ that $e_\mathcal{X}(\Gamma, A)$ must correspond to a subgroup $\mathcal{X}\mathrm{Der}(\Gamma, A) \subseteq \mathrm{Der}(\Gamma, A)$ containing $\mathrm{PDer}(\Gamma, A)$. In this paper, we introduce and study the group $\mathcal{X}\mathrm{Der}(\Gamma, A)$.

Let us be more precise. Let $\mathcal{E} : 0 \to \mathfrak{g} \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0$ be the augmentation sequence of Γ. Assume that the trivial Γ-module $\mathbb{Z}$ admits an $\mathcal{X}$-precover (see Definition 2.5 below). Consider the $\mathcal{X}$-proper extension $\xi : 0 \to L \to X_0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ with $X_0 \in \mathcal{X}$. Since $X_0 \to \mathbb{Z}$ is an $\mathcal{X}$-precover and $\mathbb{Z}\Gamma$ is projective and hence in $\mathcal{X}$, we deduce that there exists a homomorphism $f_2 : \mathfrak{g} \to L$. By the terminology of [24, p 204], $\xi = f_2\mathcal{E}$ (see 2.9 below). A derivation $d \in \mathrm{Der}(\Gamma, A)$ will be called an $\mathcal{X}$-derivation (from Γ to A), if there exists $h \in \mathrm{Hom}_\Gamma(L, A)$ and $d_0 \in \mathrm{PDer}(\Gamma, A)$, such that $d = hf_2\sigma + d_0$, where $\sigma \in \mathrm{Der}(\Gamma, \mathfrak{g})$ is defined by $\sigma(x) = x - 1$, for all $x \in \Gamma$. It is obvious that, the set of all $\mathcal{X}$-derivations from Γ to A forms a subgroup of $\mathrm{Der}(\Gamma, A)$. This subgroup is denoted by $\mathcal{X}\mathrm{Der}(\Gamma, A)$. We will show that the group $\mathcal{X}\mathrm{Der}(\Gamma, A)$ is independent of the choice of the $\mathcal{X}$-proper extension of $\mathbb{Z}$ and of the choice of the Γ-homomorphism f_2. Also there exists an isomorphism $e_\mathcal{X}(\Gamma, A) \cong \mathcal{X}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A)$. As application, we investigate the behavior of this subgroup of derivations, when $\mathcal{X} = \mathcal{F}$ is the class of flat Γ-modules and when $\mathcal{X} = \mathcal{GP}$ is the class of Gorenstein projective Γ-modules. It is shown that Γ is a trivial group if and only if for any Γ-module A, $\mathcal{F}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$ and is a finite group if and only if for any Γ-module A, $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$. An example also will be given of a group Γ and a Γ-module A, such that $\mathrm{PDer}(\Gamma, A)$ is a proper subgroup of $\mathcal{G}\mathrm{Der}(\Gamma, A)$ and the latter is a proper subgroup of $\mathrm{Der}(\Gamma, A)$.

Throughout the paper, Γ is a group and $\mathbb{Z}\Gamma$ is its associated integral group ring. All considered modules - if not specified otherwise - are left $\mathbb{Z}\Gamma$-modules that, for simplicity,

will be called (left) Γ-modules. The tensor product and Hom functor over $\mathbb{Z}\Gamma$ are denoted by $-\otimes_\Gamma -$ and $\mathrm{Hom}_\Gamma(-,-)$.

2. Preliminaries

Recall that for any group Γ, the cohomological dimension of Γ, $\mathrm{cd}\Gamma$, is the projective dimension of $\mathbb{Z}$ as a (left) trivial $\mathbb{Z}\Gamma$-module, and the homological dimension of Γ, $\mathrm{hd}\Gamma$, is the flat dimension of $\mathbb{Z}$ as a left trivial $\mathbb{Z}\Gamma$-module. Virtual cohomological dimension is a refinement of the cohomological dimension, applying to virtually torsion-free groups. Γ is virtually torsion-free if it has a torsion-free subgroup of finite index. By Serre's theorem [12, Theorem VIII.3.1] all such subgroups have the same cohomological dimension, and this common dimension is called the virtual cohomological dimension of Γ, denoted $\mathrm{vcd}\Gamma$.

Definition 2.1. A totally acyclic complex of projectives, is an exact complex

$$\cdots \to T_2 \to T_1 \to T_0 \to T_{-1} \to T_{-2} \to \cdots$$

of projective Γ-modules, which remains exact after applying the functor $\mathrm{Hom}_\Gamma(-,P)$, for any projective module P. A Γ-module G is called Gorenstein projective if it is a syzygy of a totally acyclic complex of projectives, i.e. $G = \mathrm{Ker}(T_i \to T_{i-1})$ for some integer i.

2.2. It is clear that every projective Γ-module is Gorenstein projective. The class of all projective Γ-modules will be denoted by $\mathcal{P}(\Gamma)$ and the class of all Gorenstein projective Γ-modules will be denoted by $\mathcal{GP}(\Gamma)$.

The class of Gorenstein projective modules was introduced and studied by Enochs and collaborators in a series of papers, as a natural generalizations of modules in Auslander's G-class to non-finitely generated ones, see e.g [15]. Since then, this concept has found many applications in commutative algebra, algebraic geometry, singularity theory, and relative homological algebra.

2.3. Gorenstein projective dimension. Using the class of modules in $\mathcal{GP}(\Gamma)$, one can assign an invariant to any non-zero Γ-module A, the Gorenstein projective dimension of A, denoted $\mathrm{Gpd}_\Gamma(A)$, to be the infimum of all non-negative integers n, such that there exists an acyclic complex

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to A \to 0,$$

in which all G_i's are Gorenstein projective. If there is no such resolution of finite length, we set $\mathrm{Gpd}_\Gamma(\mathrm{A}) = \infty$. Also, we set $\mathrm{Gpd}_\Gamma 0 = -\infty$. Since $\mathcal{P}(\Gamma) \subseteq \mathcal{GP}(\Gamma)$, it is plain that $\mathrm{Gpd}_\Gamma A \leq \mathrm{pd}_\Gamma A$. It is known that we have equality if $\mathrm{pd}_\Gamma A < \infty$.

Remark 2.4. The invariant $\mathrm{Gpd}_{\mathbb{Z}\Gamma}\mathbb{Z}$, which is called the Gorenstein cohomological dimension of Γ and denoted by $\mathrm{Gcd}\Gamma$, has been introduced and studied extensively in [2]. This invariant has also been studied in [5].

Definition 2.5. For a given class $\mathcal{X}$ of (left) Γ-modules, a Γ-homomorphism $\phi : X \to A$ with $X \in \mathcal{X}$ is said to be an $\mathcal{X}$-precover of A (or simply precover if $\mathcal{X}$ is understood) if the induced homomorphism $\mathrm{Hom}_\Gamma(X', X) \to \mathrm{Hom}_\Gamma(X', A)$ is surjective for all $X' \in \mathcal{X}$. $\mathcal{X}$ is said to be precovering, if every Γ-module A admits an $\mathcal{X}$-precover. For a good source for studying covers with respect to various classes of modules, see [15].

For the proof of the following result one can consult [19, Theorem 2.10].

Proposition 2.6. *Assume that A is a Γ-module and $\mathrm{Gpd}_\Gamma A$ is finite. Then there exists a short exact sequence $0 \to L \to G \to A \to 0$ of Γ-modules in which $G \to A$ is a $\mathcal{GP}$-precover of A and $\mathrm{pd}_\Gamma L < \infty$.*

It is shown in [2] that for any virtually torsion-free group Γ, $\mathrm{Gcd}\Gamma \leq \mathrm{vcd}\Gamma$ with equality if $\mathrm{vcd}\Gamma$ is finite. So the above proposition, in particular, implies that if Γ is a group of finite virtual cohomological dimension, then $\mathbb{Z}$ admits a $\mathcal{GP}$-precover.

Theorem 2.7. *Let Γ be a group of finite Gorenstein cohomological dimension. Then $\mathcal{GP}$ is a covering class, that is, any Γ-module A has a $\mathcal{GP}$-precover.*

Proof. Assume that A is an arbitrary Γ-module. In view of the above proposition, we just need to show that $\mathrm{Gpd}_\Gamma A$ is finite. This follows from [2, Proposition 2.11]. □

2.8. Relative Cohomology. Assume that $\mathcal{X}$ is precovering. In this case, for any Γ-module A there is a complex

$$X^\bullet : \ \cdots \to X_1 \to X_0 \to A \to 0$$

in which $X_i \in \mathcal{X}$ and the complex remains exact after applying the functor $\mathrm{Hom}_\Gamma(X', -)$, for any $X' \in \mathcal{X}$. $X^\bullet$ is called a (left) $\mathcal{X}$-proper resolution of A. Since we have assumed that the class $\mathcal{X}$ contains all the projective Γ-modules, every $\mathcal{X}$-proper resolution will

be exact. Known theorems from classical homological algebra, tell us that the deleted complex $\cdots \to X_1 \to X_0 \to 0$ is unique up to homotopy and so can be used to compute derived functors. We will use the notation $\mathrm{Ext}^n_{\mathcal{X}}(A, B)$ for the nth right derived functor of $\mathrm{Hom}_\Gamma(-, B)$ using an $\mathcal{X}$-proper resolution of A. If $A = \mathbb{Z}$, then $\mathrm{Ext}^n_{\mathcal{X}}(\mathbb{Z}, -)$ is defined to be the nth relative cohomology group and will be denoted by $H^n_{\mathcal{X}}(\Gamma, -)$.

Now choose a projective resolution $P_\bullet \longrightarrow \mathbb{Z}$ over $\mathbb{Z}\Gamma$. So there exists a morphism of complexes $\varphi : P_\bullet \longrightarrow X_\bullet$ lifting the identity morphism of $\mathbb{Z}$. By applying the functor $\mathrm{Hom}_\Gamma(-, B)$, for any Γ-module B, we get a comparison homomorphism $\varepsilon^n_{\mathcal{X}}(B) : H^n_{\mathcal{X}}(\Gamma, B) \to H^n(\Gamma, B)$, from relative cohomology to absolute ones. In fact, $\varepsilon^n_{\mathcal{X}} = \mathrm{H}^n\mathrm{Hom}_\Gamma(\varphi, B)$.

Since the functor $\mathrm{Hom}_\Gamma(-, B)$ is a left exact functor, one can see easily that $\varepsilon^0_{\mathcal{X}}(B) : H^0_{\mathcal{X}}(\Gamma, B) \to H^0(\Gamma, B)$ is an isomorphism. In [1, Proposition 2.1], it is shown that $\mathrm{Ext}^1_{\mathcal{X}}(A, B)$ is a subgroup of $\mathrm{Ext}^1_\Gamma(A, B)$, for all Γ-modules A and B. Hence $\varepsilon^1_{\mathcal{X}} : H^1_{\mathcal{X}}(\Gamma, B) \to H^1(\Gamma, B)$ is a monomorphism.

2.9. Assume that $\varepsilon : 0 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow C \longrightarrow 0$ is a short exact sequence of Γ-modules and $\alpha : E_1 \longrightarrow A$ is a Γ-homomorphism. We denote the pushout of the extension ε along α by $\alpha\varepsilon$. In addition, its equivalence class will be denoted by $[\alpha\varepsilon]$, for details see [23, Chapter VII].

2.10. Derivations. Let R be a commutative ring with identity, Λ a noetherian R-algebra and let A be a Λ-bimodule. An R-linear map $d : \Lambda \longrightarrow A$ is said to be a *derivation*, provided that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \Lambda$. It is easily seen that the set of all derivations from Λ to A, denoted by $\mathrm{Der}(\Lambda, A)$, is a submodule of R-linear maps from Λ to A. A derivation $d \in \mathrm{Der}(\Lambda, A)$ is said to be *principal* (or *inner*), if there exists an object $a \in A$ such that for any $x \in \Lambda$, $d(x) = ax - xa$. It is known that the set of all principal derivatons, $\mathrm{PDer}(\Lambda, A)$, is a subgroup of $\mathrm{Der}(\Lambda, A)$. Now assume that $\Lambda = \mathbb{Z}\Gamma$ and A is a left Γ-module. Considering A as a right Γ-module by trivial action, it will be a bimodule. So, in this case, the above definition can be simplified as follows:

A map $d : \Gamma \longrightarrow A$ is said to be a derivation from Γ to A, if $d(xy) = d(x) + xd(y)$ for all $x, y \in \Gamma$. The derivation d is called principal if there exists $a \in A$ such that for all $x \in \Gamma$, $d(x) = xa - a$. The group of derivations and principal derivations from Γ to

A are denoted by $\mathrm{Der}(\Gamma, A)$ and $\mathrm{PDer}(\Gamma, A)$, respectively. Now let us explain that the derivation functor is represented by the augmentation ideal.

Let $\mathcal{E} : 0 \to \mathfrak{g} \to \mathbb{Z}\Gamma \xrightarrow{\epsilon} \mathbb{Z} \to 0$ be the augmentation sequence of Γ. We recall that the augmentation ideal $\mathfrak{g}$ of $\mathbb{Z}\Gamma$ is the kernel of the augmentation homomorphism $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ which is defined by $\epsilon(\Sigma a_i x_i) = \Sigma a_i$, $a_i \in \mathbb{Z}$, $x_i \in \Gamma$. Also fix the derivation $\sigma : \Gamma \to \mathfrak{g}$ defined by $\sigma(x) = x - 1$. It is well-known [24, Theorem 5.21] that, for any derivation $d : \Gamma \to A$, there exists a Γ-homomorphism $h \in \mathrm{Hom}_\Gamma(\mathfrak{g}, A)$ such that $d = h\sigma$. Indeed, the map $\phi : \mathrm{Hom}_\Gamma(\mathfrak{g}, A) \longrightarrow \mathrm{Der}(\Gamma, A)$, sending each morphism$h$ to $h\sigma$, is a natural isomorphism of abelian groups; see also [26, Lemma 6.4.4].

3. $\mathcal{X}$-DERIVATIONS

We begin this section by recalling the notion of $\mathcal{X}$-proper extensions of $\mathbb{Z}$ by a Γ-module A. The extension $0 \to A \to E \to \mathbb{Z} \to 0$, which is $\mathrm{Hom}_\Gamma(X, -)$-exact, for $X \in \mathcal{X}$, is called an $\mathcal{X}$-proper extension of $\mathbb{Z}$ by A. The group of all equivalence classes of $\mathcal{X}$-proper extensions of $\mathbb{Z}$ by A is denoted by $e_{\mathcal{X}}(\Gamma, A)$. Now, consider an $\mathcal{X}$-precover $\pi : X_0 \to \mathbb{Z}$ of $\mathbb{Z}$. Since $X_0 \to \mathbb{Z}$ is a precover and $\mathbb{Z}\Gamma$ belongs to $\mathcal{X}$, there exists homomorphism $f_1 : \mathbb{Z}\Gamma \to X_0$ commuting the square on the right. Hence we get f_2 and therefore a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{i} & \mathbb{Z}\Gamma & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle id}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & X_0 & \xrightarrow{\pi} & \mathbb{Z} & \longrightarrow & 0, \end{array}$$

in which $L = \mathrm{Ker}\pi$.

Definition 3.1. A derivation $d \in \mathrm{Der}(\Gamma, A)$ is called an $\mathcal{X}$-derivation if there exists $h \in \mathrm{Hom}_\Gamma(L, A)$ and $d_0 \in \mathrm{PDer}(\Gamma, A)$, such that $d = hf_2\sigma + d_0$. It is obvious that the set of all $\mathcal{X}$-derivations from Γ to A, forms a subgroup of $\mathrm{Der}(\Gamma, A)$. This subgroup is denoted by $\mathcal{X}\mathrm{Der}(\Gamma, A)$.

The result below reveals that, our definition of $\mathcal{X}$-derivations is independent of the induced morphism f_2 as well as the choice of the $\mathcal{X}$-precover $X_0 \to \mathbb{Z}$.

Lemma 3.2. *The group $\mathcal{X}\mathrm{Der}(\Gamma, A)$ is independent of the choice of the $\mathcal{X}$-precover $X_0 \to \mathbb{Z}$ and of the choice of the Γ-homomorphism f_2.*

Proof. Let $f_2' : \mathfrak{g} \to L$ be an another homomorphism that commutes the above diagram. A relative version of the comparison lemma [15, Sec. 8.1] implies that there exists a Γ-homomorphism $\theta : \mathbb{Z}\Gamma \to L$ such that $f_2' - f_2 = \theta i$. Thus for any $h \in \mathrm{Hom}_\Gamma(L, A)$ we have $h(f_2' - f_2)\sigma = h\theta i\sigma \in \mathrm{PDer}(\Gamma, A)$. Therefore $hf_2'\sigma = hf_2\sigma + d_0 \in \mathcal{X}\mathrm{Der}(\Gamma, A)$, where $h\theta i\sigma = d_0 \in \mathrm{PDer}(\Gamma, A)$. This implies that our definition is independent of the induced map f_2.

Next assume that $0 \to L' \to X_0' \to \mathbb{Z} \to 0$ is another $\mathcal{X}$-proper extension of $\mathbb{Z}$ with $X_0' \in \mathcal{X}$. We have the following commutative diagrams:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g} & \xrightarrow{i} & \mathbb{Z}\Gamma & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & {\scriptstyle f_2'}\downarrow & & {\scriptstyle f_1'}\downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & L' & \xrightarrow{u'} & X_0' & \xrightarrow{\pi'} & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g} & \xrightarrow{i} & \mathbb{Z}\Gamma & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{u} & X_0 & \xrightarrow{\pi} & \mathbb{Z} & \longrightarrow & 0 \\
 & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_1}\downarrow & & {\scriptstyle id}\downarrow & & \\
0 & \longrightarrow & L' & \xrightarrow{u'} & X_0' & \xrightarrow{\pi'} & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Since f_2' and g_2f_2 are lifted by the identity map over $\mathbb{Z}$, for any $h' \in \mathrm{Hom}_\Gamma(L', A)$, we have $h'(f_2' - g_2f_2)\sigma \in \mathrm{PDer}(\Gamma, A)$. Now, since $h'g_2 \in \mathrm{Hom}_\Gamma(L, A)$, this implies that $h'f_2'\sigma \in \mathcal{X}\mathrm{Der}(\Gamma, A)$. This completes the proof. □

The result below is well-known from the point of view of general homological algebra and we include its proof only for the sake of completeness.

Lemma 3.3. *Let $\mathcal{E} : 0 \to \mathfrak{g} \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0$ be the augmentation sequence of Γ and $\xi : 0 \to L \to X_0 \to \mathbb{Z} \to 0$ be an $\mathcal{X}$-proper extension of $\mathbb{Z}$ with $X_0 \in \mathcal{X}$. Then*

- (*i*) *Any equivalence class of extensions of $\mathbb{Z}$ by A is of the form $[h\mathcal{E}]$, where $h : \mathfrak{g} \to A$ is a Γ-homomorphism.*
- (*ii*) *Any equivalence class of $\mathcal{X}$-proper extensions of $\mathbb{Z}$ by A is of the form $[h'\xi]$, where $h' : L \to A$ is a Γ-homomorphism.*

Proof. Let us prove (*ii*). The proof of (*i*) is similar. Let $\varepsilon : 0 \to A \to E \to \mathbb{Z} \to 0$ be an extension of $\mathbb{Z}$ by A such that $\varepsilon \in [h'\xi]$, for some Γ-homomorphism $h' : L \to A$. Then we

have the following commutative diagram

$$\begin{array}{ccccccccc} \xi: 0 & \longrightarrow & L & \longrightarrow & X_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle id}\downarrow & & \\ \varepsilon: 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Since $X_0 \to \mathbb{Z}$ is an $\mathcal{X}$-precover, the upper row remains exact after applying the functor $\mathrm{Hom}_\Gamma(X', -)$, for any $X' \in \mathcal{X}$. This leads us to infer that the sequence

$$0 \to \mathrm{Hom}_\Gamma(X', A) \to \mathrm{Hom}_\Gamma(X', E) \to \mathrm{Hom}_\Gamma(X', \mathbb{Z}) \to 0$$

is exact, i.e. ε is an $\mathcal{X}$-proper extension of $\mathbb{Z}$ by A. Now suppose that $\varepsilon: 0 \to A \to E \to \mathbb{Z} \to 0$ is an $\mathcal{X}$-proper extension of $\mathbb{Z}$ by A. Since $X_0 \in \mathcal{X}$, there exist Γ-homomorphisms ϕ and h' such that the following diagram commutes

$$\begin{array}{ccccccccc} \xi: 0 & \longrightarrow & L & \longrightarrow & X_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle h'}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle id}\downarrow & & \\ \varepsilon: 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Consequently, $[\varepsilon] = [h'\xi]$ and so the proof is completed. □

Now, we are ready to prove the main result of this paper.

Theorem 3.4. $e_{\mathcal{X}}(\Gamma, A) \cong \mathcal{X}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A)$. *In particular,* $H^1_{\mathcal{X}}(\Gamma, A) \cong \mathcal{X}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A)$.

Proof. Pick an $\mathcal{X}$-proper sequence

$$0 \to L \xrightarrow{u} X_0 \xrightarrow{\pi} \mathbb{Z} \to 0,$$

with $X_0 \in \mathcal{X}$. Apply $\mathrm{Hom}_\Gamma(-, A)$ to this sequence to get the exact sequence

$$0 \to \mathrm{Hom}_\Gamma(\mathbb{Z}, A) \to \mathrm{Hom}_\Gamma(X_0, A) \xrightarrow{\vartheta} \mathrm{Hom}_\Gamma(L, A) \to \mathrm{Coker}\vartheta \to 0.$$

By Lemma 3.3, one can easily show that $\theta: e_{\mathcal{X}}(\Gamma, A) \to \mathrm{Hom}_\Gamma(L, A)/\mathrm{Im}\vartheta$, defined by $\theta([h\xi]) = h + \mathrm{Im}\vartheta$ is an isomorphism. On the other hand, it is easy to check that the map

$$\varphi: \mathrm{Hom}_\Gamma(L, A)/\mathrm{Im}\vartheta \to \mathcal{X}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A),$$

defined by $\varphi(h + \mathrm{Im}\vartheta) = hd_2 + \mathrm{PDer}(\Gamma, A)$, where $d_2 = f_2\sigma$, is onto.

Suppose $hd_2 \in \mathrm{PDer}(\Gamma, A)$. Thus there exists a Γ-homomorphism $\alpha : \mathbb{Z}\Gamma \to A$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & \mathbb{Z}\Gamma \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle \alpha} \\ L & \xrightarrow{h} & A, \end{array}$$

is commutative. Since the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & \mathbb{Z}\Gamma \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ L & \xrightarrow{u} & X_0 \end{array}$$

is the pushout diagram, there exists a unique Γ-homomorphism $\theta : X_0 \to A$ making related triangles commute. Therefore $h \in \mathrm{Im}\vartheta$, and so, φ is an isomorphism. Hence, we obtain the isomorphism $e_{\mathcal{X}}(\Gamma, A) \cong \mathcal{X}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A)$. Now the proof can be completed just by noting the fact that $e_{\mathcal{X}}(\Gamma, A) \cong H^1_{\mathcal{X}}(\Gamma, A)$. □

For a given class $\mathcal{X}$ of Γ-modules, we let ${}^{\perp}\mathcal{X}$ (respectively, $\mathcal{X}^{\perp}$) to be the class of all Γ-modules A such that $\mathrm{Ext}^1_{\Gamma}(A, X) = 0$ (respectively, $\mathrm{Ext}^1_{\Gamma}(X, A) = 0$), for all $X \in \mathcal{X}$. ${}^{\perp}\mathcal{X}$ and $\mathcal{X}^{\perp}$ are called left and right orthogonal classes of $\mathcal{X}$.

Corollary 3.5. *If $A \in \mathcal{X}^{\perp}$, then $\mathcal{X}\mathrm{Der}(\Gamma, A) = \mathrm{Der}(\Gamma, A)$.*

Proof. Since $A \in \mathcal{X}^{\perp}$, all short exact sequences $0 \to A \to E \to \mathbb{Z} \to 0$ of Γ-modules are $\mathcal{X}$-proper. Hence $e_{\mathcal{X}}(\Gamma, A) = e(\Gamma, A)$. This, in view of the above theorem, implies the result. □

Corollary 3.6. *If A is a trivial Γ-module, then*

$$e_{\mathcal{X}}(\Gamma, A) \cong \mathcal{X}\mathrm{Der}(\Gamma, A) \cong \{\alpha \in \mathrm{Hom}_{\mathrm{Group}}(\Gamma/[\Gamma, \Gamma], A) \mid \alpha = hf_2\bar{\sigma} \text{ for some } h \in Hom_{\Gamma}(L, A)\},$$

where $[\Gamma, \Gamma]$ is the derived group of Γ and $\bar{\sigma}([\Gamma, \Gamma]x) = x - 1$ for all $x \in \Gamma$.

Proof. By [26, Corollary 6.4.6], $H^1(\Gamma, A) \cong \mathrm{Der}(\Gamma, A) \cong \mathrm{Hom}_{\mathrm{Group}}(\Gamma, A)$. Since A is an abelian group, for any $d \in \mathrm{Der}(\Gamma, A)$, $d([x, y]) = 0$ for all $[x, y] \in [\Gamma, \Gamma]$. Therefore $\mathrm{Der}(\Gamma, A) \cong \mathrm{Hom}_{\mathrm{Group}}(\Gamma/[\Gamma, \Gamma], A)$. Hence the result follows from the definition of $\mathcal{X}\mathrm{Der}(\Gamma, A)$ and using the fact that $\mathrm{PDer}(\Gamma, A) = 0$. □

Let $\mathcal{X} = \mathcal{F}$ be the class of all flat Γ-modules. It is known that the class $\mathcal{F}$ is precovering (see [9]). Consequently $\mathbb{Z}$ admits an $\mathcal{F}$-(pre)cover. In this case $\mathcal{F}$-derivations will be denoted by $\mathcal{F}\mathrm{Der}(\Gamma, A)$.

3.7. Note that by Theorem 3.4, for any Γ-module A,

$$e_{\mathcal{F}}(\Gamma, A) \cong \mathcal{F}\mathrm{Der}(\Gamma, A)/\mathrm{PDer}(\Gamma, A) \cong H^1_{\mathcal{F}}(\Gamma, A).$$

A Γ-module A is called cotorsion if $A \in \mathcal{F}^{\perp}$. That is, $\mathrm{Ext}^1_{\Gamma}(F, A) = 0$, for all flat Γ-module F. The following corollary follows directly from Corollary 3.5.

Corollary 3.8. *If A is a cotorsion Γ-module, then $\mathcal{F}\mathrm{Der}(\Gamma, A) = \mathrm{Der}(\Gamma, A)$.*

Corollary 3.9. *Γ is a trivial group if and only if for any Γ-module A, $\mathcal{F}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$.*

Proof. $\mathcal{F}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$ if and only if $H^1_{\mathcal{F}}(\Gamma, A) = 0$, which is equivalent to say that $\mathrm{hd}\Gamma = 0$. This, in turn, is equivalent to say that Γ is trivial, see [10, Proposition 4.12]. □

Remark 3.10. Let $\mathbb{Z}\Gamma$ be a coherent ring, i.e. every finitely generated (left) ideal of $\mathbb{Z}\Gamma$ is finitely presented. Assume that $\mathcal{GF}$ is the class of all Gorenstein flat Γ-modules. Then $\mathcal{GF}$ is a precovering class. In this case, we have a subgroup of $\mathrm{Der}(\Gamma, A)$ consisting of $\mathcal{GF}$-derivations, denoted $\mathcal{GF}\mathrm{Der}(\Gamma, A)$.

Recall that a Γ-module M is said to be Gorenstein flat, if there exists an acyclic complex of flat Γ-modules: $\mathbf{F}_\bullet : \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $M = \ker(F_0 \to F_{-1})$ and for any right injective Γ-module I, the functor $I \otimes_{\Gamma} -$ leaves the complex $\mathbf{F}_\bullet$ exact. Such a complex $\mathbf{F}_\bullet$ is called a complete flat resolution of M, see [16].

Now assume that Γ is a finite group. According to Proposition 4.12 of [3], $\mathbb{Z}$ is a Gorenstein flat Γ-module. This, in particular, implies that $\mathcal{GF}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$.

We finish this remark by giving some situations in which $\mathbb{Z}\Gamma$ is a coherent ring. Let Γ be a finitely generated soluble group. Then $\mathbb{Z}\Gamma$ is coherent, if one of the following conditions hold (see [11]):

(a) Γ is coherent, that is any finitely generated subgroup of Γ is finitely presented;

(b) Γ is polycyclic or is an ascending HNN-group over a polycyclic base group.

4. Gorenstein derivations

In this section we consider the special case where $\mathcal{X} = \mathcal{GP}(\Gamma)$ is the class of all Gorenstein projective Γ-modules. We assume that Γ is of finite Gorenstein cohomological dimension. Note that by Theorem 2.7, this implies that $\mathcal{GP}$ is a precovering class. In this case the group of $\mathcal{X}$-derivations will be denoted by $\mathcal{G}\mathrm{Der}(\Gamma, A)$ and its elements are called Gorenstein derivations.

4.1. Let Γ be a finite group. By invoking [12, Proposition VI.2.6], in view of [12, Proposition VI.3.5], we obtain that the trivial Γ-module $\mathbb{Z}$ is Gorenstein projective. Hence $id : \mathbb{Z} \to \mathbb{Z}$ is an $\mathcal{GP}$-precover of $\mathbb{Z}$. So by the notations of the Definition 3.1, we then have $L = 0$ and hence $f_2 = 0$. Therefore, $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$.

Proposition 4.2. *Γ is a finite group if and only if for any Γ-module A, $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$.*

Proof. Assume that for any Γ-module A, $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{PDer}(\Gamma, A)$. This in view of Theorem 3.4, implies that $H^1_{\mathcal{GP}}(\Gamma, A) = \mathrm{Ext}^1_{\mathcal{GP}}(\mathbb{Z}, A) = 0$, for any Γ-module A. So any $\mathcal{GP}$-proper extension of $\mathbb{Z}$ is split. In particular

$$0 \to L \to G \to \mathbb{Z} \to 0,$$

with $G \in \mathcal{GP}$ is split. Hence $\mathbb{Z}$, being a direct summand of G, is Gorenstein projective. Now the result follows from [2, Proposition 2.19]. The converse has been shown in 4.1. □

Remark 4.3. It is worth noting that the problem finding suitable conditions under which for a given algebra Λ, every derivation is principal, has been the subject of some expositions; see for example [8, 7, 13, 14]. In the following two results, we provide some criteria under which every derivation is a Gorenstein derivation.

If Γ is a finite group, then $H^1(\Gamma, \mathbb{Z}) = 0$, see e.g. [26, Exercise 6.1.5(2)]. So $\mathrm{Der}(\Gamma, \mathbb{Z}) = \mathrm{PDer}(\Gamma, \mathbb{Z})$. On the other hand, by the above proposition, for any finite group Γ, $\mathrm{PDer}(\Gamma, \mathbb{Z}) = \mathcal{G}\mathrm{Der}(\Gamma, \mathbb{Z})$. Therefore we have proved the following corollary.

Corollary 4.4. *Let Γ be a finite group. Then $\mathcal{G}\mathrm{Der}(\Gamma, \mathbb{Z}) = \mathrm{Der}(\Gamma, \mathbb{Z})$*

Proposition 4.5. *Let Γ be a group and A be a Γ-module. Then $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{Der}(\Gamma, A)$ if one of the following conditions holds:*

(*i*) $\mathrm{cd}\Gamma < \infty$.

(*ii*) $\mathrm{pd}_\Gamma A < \infty$.

(*iii*) $\mathrm{id}_\Gamma A < \infty$.

Proof. (*i*). Let $\mathrm{cd}\Gamma < \infty$. Then the augmentation map $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ is an $\mathcal{GP}$-precover of $\mathbb{Z}$, because $\mathrm{pd}_\Gamma\mathbb{Z}$ and hence $\mathrm{Gpd}_\Gamma\mathbb{Z}$ is finite. So we can consider $L = \mathfrak{g}$, $f_1 = id_{\mathbb{Z}\Gamma}$ and $f_2 = id_{\mathfrak{g}}$. Hence $\mathrm{Hom}_\Gamma(L, A)f_2 = \mathrm{Hom}_\Gamma(\mathfrak{g}, A)$. On the other hand, by [24, Theorem 5.21], $\mathrm{Hom}_\Gamma(\mathfrak{g}, A) \cong \mathrm{Der}(\Gamma, A)$. Therefore $\mathcal{G}\mathrm{Der}(\Gamma, A) = \mathrm{Der}(\Gamma, A)$.

(*ii*) and (*iii*). By [15, Remark 10.2.2] and [18, Theorem 3.8 (i)], if $\mathrm{pd}_\Gamma A < \infty$ or $\mathrm{id}_\Gamma A < \infty$, then $A \in \mathcal{GP}(\Gamma)^\perp$. Hence we obtain the result from Corollary 3.5. □

4.6. We recall that a finite p-group Γ, for some prime integer p, is Frattinian if $Z(\Gamma) \neq Z(M)$ for all maximal subgroups M of Γ. Here $Z(\Gamma)$ is the center of Γ, that is, $Z(\Gamma) = \{x \in \Gamma \mid xy = yx \text{ for all } y \in \Gamma\}$. Γ is called strongly Frattinian, if $C_\Gamma(Z(\phi(\Gamma))) = \phi(\Gamma)$, where $C_\Gamma(Z(\phi(\Gamma))) = \{x \in \Gamma \mid xy = yx \text{ for all } y \in Z(\phi(\Gamma))\}$. The Frattini subgroup of an arbitrary group Γ, denoted by $\phi(\Gamma)$, is defined to be the intersection of all the maximal subgroups, with the stipulation that it shall equal Γ if Γ has no maximal subgroups. It is easy to see that under conjugation by the elements of Γ, $Z(\phi(\Gamma))$ becomes a $\Gamma/\phi(\Gamma)$-module, where $\phi(\Gamma)$ and $Z(\phi(\Gamma))$ are Frattini subgroup and the center of Frattini subgroup of Γ, respectively. Now, let Γ be a non-abelian finite p-group of order p^3. It is not hard to check that Γ is Frattinian but it is not strongly Frattinian. Since $\Gamma/\phi(\Gamma)$ is a finite group, as we saw above, $\mathcal{G}\mathrm{Der}(\Gamma/\phi(\Gamma), Z(\phi(\Gamma))) = \mathrm{PDer}(\Gamma/\phi(\Gamma), Z(\phi(\Gamma)))$. On the other hand, by [25, Proposition 2], $H^1(\Gamma/\phi(\Gamma), Z(\phi(\Gamma))) \neq 0$. Using this fact, in view of [26, Theorem 6.4.5], one may deduce that the group $\mathcal{G}\mathrm{Der}(\Gamma/\phi(\Gamma), Z(\phi(\Gamma)))$ is a proper subgroup of $\mathrm{Der}(\Gamma/\phi(\Gamma), Z(\phi(\Gamma)))$.

Lemma 4.7. *Let* H *be a subgroup of* Γ *of finite index. Suppose that* $G \to \mathbb{Z}$ *is a* $\mathcal{GP}$*-precover of* $\mathbb{Z}$ *over* $\mathbb{Z}\Gamma$*. Then* $G \to \mathbb{Z}$ *is also a* $\mathcal{GP}$*-precover of* $\mathbb{Z}$ *as* $\mathbb{Z}$H*-module.*

Proof. Since H is a subgroup of finite index of Γ, for any H-module A, $\mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}\Gamma, A) \cong \mathbb{Z}\Gamma \otimes_{\mathrm{H}} A$, see [12, Proposition III 5.9]. This fact shows that if $P \in \mathcal{P}(\mathrm{H})$ then $\mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}\Gamma, P) \in \mathcal{P}(\Gamma)$. By using this fact and the adjointness of Hom and $\otimes$, we may obtain that $\mathcal{GP}(\Gamma) \subseteq \mathcal{GP}(\mathrm{H})$. On the other hand, it is easy to show that if $G' \in \mathcal{GP}(\mathrm{H})$, then $\mathbb{Z}\Gamma \otimes_{\mathrm{H}} G' \in \mathcal{GP}(\Gamma)$, (see [5, Example 2.1 (c)]). Let G' be a Gorenstein projective H-module. Another use of

the adjointness of Hom and $\otimes$ implies we have the following commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{H}}(G', G) & \longrightarrow & \mathrm{Hom}_{\mathrm{H}}(G', \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \\
\mathrm{Hom}_{\mathrm{H}}(G', \mathrm{Hom}_{\Gamma}(\mathbb{Z}\Gamma, G)) & \longrightarrow & \mathrm{Hom}_{\mathrm{H}}(G', \mathrm{Hom}_{\Gamma}(\mathbb{Z}\Gamma, \mathbb{Z})) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \\
\mathrm{Hom}_{\Gamma}(\mathbb{Z}\Gamma \otimes_{\mathrm{H}} G', G) & \longrightarrow & \mathrm{Hom}_{\Gamma}(\mathbb{Z}\Gamma \otimes_{\mathrm{H}} G', \mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

Since $\mathbb{Z}\Gamma \otimes_{\mathrm{H}} G' \in \mathcal{GP}(\Gamma)$ and $G \longrightarrow \mathbb{Z}$ is a $\mathcal{GP}$-precover of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, the lower row is exact. Therefore the upper row is also exact. So the proof is complete. □

Remark 4.8. We should remark that, as we have seen, the proof of the above lemma relies on the fact that for any H-module A, there is a Γ-isomorphism $\mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}\Gamma, A) \cong \mathbb{Z}\Gamma \otimes_{\mathrm{H}} A$. It is worth to mention that the argument works in a slightly more general situation. Namely, assume that $\phi : R \longrightarrow S$ is a ring extension such that S is a finitely generated projective R-module and $\mathrm{Hom}_R(S, R)$ is a projective S-module. Then, if $G \to M$ is a $\mathcal{GP}$-precover of a given S-module M, it will be also a $\mathcal{GP}$-precover of M as an R-module.

Theorem 4.9. *Let* H *be a subgroup of* Γ *of finite index. Let* A *be a* H*-module. Then* $\mathcal{G}\mathrm{Der}(\mathrm{H}, A) = \mathcal{G}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma} A)$. *In particular, if* H *is a torsion-free subgroup of* Γ *and* $\mathrm{vcd}\Gamma < \infty$, *then* $\mathrm{Der}(\mathrm{H}, A) = \mathcal{G}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma} A)$.

Proof. Let $0 \to L \to G \to \mathbb{Z} \to 0$ be a $\mathcal{GP}$-proper extension of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ with $G \in \mathcal{GP}(\Gamma)$. By the above lemma, this can also be regarded as a $\mathcal{GP}$-proper extension of $\mathbb{Z}$ over $\mathbb{Z}\mathrm{H}$ with $G \in \mathcal{GP}(\mathrm{H})$. We consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}' & \xrightarrow{i} & \mathbb{Z}\mathrm{H} & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathbb{Z}\Gamma & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & L & \xrightarrow{u} & G & \xrightarrow{\pi} & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where i_1 and i_2 are inclusion maps. Now let $d' \in \mathcal{G}\mathrm{Der}(\mathrm{H}, A)$. By definition, $d' = h' f_2' \sigma' + d_0'$, where $h' : L \to A$ is an H-homomorphism, $f_2' = f_2 i_2$ and $d_0' \in \mathrm{PDer}(\mathrm{H}, A)$.

Since $\mathrm{Hom}_{\mathrm{H}}(L, A) \cong \mathrm{Hom}_{\Gamma}(L, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$, corresponding to h', there exists a unique Γ-homomorphism $h : L \to \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A$. Since $d'_0 \in \mathrm{PDer}(\mathrm{H}, A)$, there is H-homomorphisms $\varphi' : \mathbb{Z}\mathrm{H} \to A$ such that $d'_0 = \varphi'_{|_{\mathfrak{g}'}}\sigma'$, where $\mathfrak{g}'$ is the augmentation ideal of $\mathbb{Z}\mathrm{H}$. Hence we have a Γ-homomorphism $\theta = id_{\mathbb{Z}\Gamma} \otimes_{\mathrm{H}} \varphi' : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma \otimes_{\mathrm{H}} A \cong \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A$. Then $d_0 = \theta_{|\mathfrak{g}}\sigma \in \mathrm{PDer}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$. Therefore $d = hf_2\sigma + d_0 \in \mathcal{G}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$. In fact, we have defined

$$\psi : \mathcal{G}\mathrm{Der}(\mathrm{H}, A) \to \mathcal{G}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A),$$

by $\psi(h'f'_2\sigma + d'_0) := hf_2\sigma + d_0$. For the converse, consider $d = hf_2\sigma + d_0 \in \mathcal{G}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$. Since $d_0 \in \mathrm{PDer}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$, there exists a Γ-homomorphism $\theta : \mathbb{Z}\Gamma \to \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A$ such that $d_0 = \theta_{|\mathfrak{g}}\sigma$. Since $\mathrm{Hom}_{\Gamma}(\mathbb{Z}\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A) \cong \mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}\Gamma, A)$, there exists an H-homomorphism $\varphi : \mathbb{Z}\Gamma \to A$. Hence $d'_0 = \varphi i_{1|\mathfrak{g}'}\sigma' \in \mathrm{PDer}(\mathrm{H}, A)$ and then $d' = hf_2i_2\sigma' + d'_0 \in \mathcal{G}\mathrm{Der}(\mathrm{H}, A)$. The second part, follows from the Proposition 4.5. □

Remark 4.10. A similar method as in the proof of the above theorem can be applied to the case $\mathcal{X} = \mathcal{F}$ to show that $\mathcal{F}\mathrm{Der}(\mathrm{H}, A) = \mathcal{F}\mathrm{Der}(\Gamma, \mathrm{Coind}_{\mathrm{H}}^{\Gamma}A)$.

Corollary 4.11. *Let* H *be a subgroup of* Γ *of finite index. Let* A *be a* Γ*-module. Then* $\mathcal{G}\mathrm{Der}(\mathrm{H}, A) = \mathcal{G}\mathrm{Der}(\Gamma, \mathbb{Z}[\Gamma/\mathrm{H}] \otimes A)$*, where* $\mathbb{Z}[\Gamma/\mathrm{H}]$ *is permutation module and* Γ *acts diagonally on the tensor product. In particular,* $\mathcal{G}\mathrm{Der}(\mathrm{H}, \mathbb{Z}) = \mathcal{G}\mathrm{Der}(\Gamma, \mathbb{Z}[\Gamma/\mathrm{H}])$.

Proof. By [12, Proposition III.5.6(a)], $\mathrm{Ind}_{\mathrm{H}}^{\Gamma}\mathrm{Res}_{\mathrm{H}}^{\Gamma}\mathrm{A} \cong \mathbb{Z}[\Gamma/\mathrm{H}] \otimes \mathrm{A}$, where $\mathrm{Ind}_{\mathrm{H}}^{\Gamma}\mathrm{Res}_{\mathrm{H}}^{\Gamma}\mathrm{A} = \mathbb{Z}\Gamma \otimes_{\mathrm{H}} \mathrm{Res}_{\mathrm{H}}^{\Gamma}\mathrm{A}$. The result hence follows from the above theorem. □

4.12. There is an exact sequence of cohomology groups connecting the absolute, the Tate and the relative cohomology groups. The sequence first was discovered by Avramov and Martsinkovsky in [4, Theorem 7.1], for finite modules of finite Gorenstein dimension over commutative rings. Then it was extended to different settings, such as non-finite modules and also complexes, by several authors, see e.g. [20]. With our notation, we may rewrite the sequence in the following form:

Let Γ be a group of finite Gorenstein cohomological dimension n. Then for any Γ-module A, there exists a long exact sequence of cohomology groups

$$\begin{aligned} 0 \longrightarrow\ & H^1_{\mathcal{GP}}(\Gamma, A) \longrightarrow H^1(\Gamma, A) \longrightarrow \widehat{H}^1(\Gamma, A) \\ \longrightarrow\ & H^2_{\mathcal{GP}}(\Gamma, A) \longrightarrow \quad \cdots \\ \longrightarrow\ & H^n_{\mathcal{GP}}(\Gamma, A) \longrightarrow H^n(\Gamma, A) \longrightarrow \widehat{H}^n(\Gamma, A) \longrightarrow 0 \end{aligned}$$

in which $H^i_{\mathcal{GP}}$ denotes the ith relative cohomology and $\widehat{H}^i$ denotes the ith Tate cohomology group.

In the following we give an example of a Γ-module A for which $\mathrm{PDer}(\Gamma, A) < \mathcal{G}\mathrm{Der}(\Gamma, A) < \mathrm{Der}(\Gamma, A)$. Here, '$<$' denotes the proper subgroup.

Example 4.13. Let Γ be a group with $\mathrm{cd}\Gamma = \infty$ and $\mathrm{vcd}\Gamma = 1$. For example, assume that Γ is a non-trivial non-free group with $\mathrm{vcd}\Gamma = 1$.

Let H be a subgroup of Γ, such that $[\Gamma : \mathrm{H}] < \infty$ and $\mathrm{cd}\mathrm{H} = 1$. Such subgroup exists, because we have assumed that $\mathrm{vcd}\Gamma = 1$. Set $A := \mathfrak{g} \oplus \mathrm{Ind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}'$, where $\mathfrak{g}'$ is the augmentation ideal of $\mathbb{Z}\mathrm{H}$ and $\mathrm{Ind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}'$ is the induction module $\mathbb{Z}\Gamma \otimes_{\mathrm{H}} \mathfrak{g}'$.

The long exact sequence of Tate cohomology groups arising from the sequence

$$0 \to \mathfrak{g} \to \mathbb{Z}\Gamma \to \mathbb{Z} \to 0,$$

in view of the fact that $\widehat{H}^i(\Gamma, \mathbb{Z}\Gamma) = 0$, for all integers i, implies that $\widehat{H}^0(\Gamma, \mathbb{Z}) \cong \widehat{H}^1(\Gamma, \mathfrak{g})$. But since $\mathrm{cd}\Gamma = \infty$, by [21, 4.2], $\widehat{H}^0(\Gamma, \mathbb{Z}) \neq 0$. Hence $\widehat{H}^1(\Gamma, A) \neq 0$.

Since $\mathrm{vcd}\Gamma = 1$, it follows from [4, Theorem 4.2] and [12, Proposition X.2.1(a)] that $H^i_{\mathcal{GP}}(\Gamma, -) = 0$, for all $i \geq 2$. So, in view of the long exact sequence of 4.12, we get the following exact sequence

$$0 \to H^1_{\mathcal{GP}}(\Gamma, A) \to H^1(\Gamma, A) \to \widehat{H}^1(\Gamma, A) \to 0.$$

We claim that $H^1(\mathrm{H}, \mathfrak{g}') \neq 0$. Assume on the contrary that $H^1(\mathrm{H}, \mathfrak{g}') = 0$. So the sequence $\mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}, \mathbb{Z}\mathrm{H}) \longrightarrow \mathrm{Hom}_{\mathrm{H}}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0$ is exact. This implies that $\mathrm{cd}\mathrm{H} = 0$. This is a contradiction. So the claim. But since $\mathrm{cd}\mathrm{H} < \infty$, $H^1(\mathrm{H}, \mathfrak{g}') = H^1_{\mathcal{GP}}(\mathrm{H}, \mathfrak{g}')$. Now the relative version of the Shapiro's lemma applies to show that $H^1_{\mathcal{GP}}(\mathrm{H}, \mathfrak{g}') \cong H^1_{\mathcal{GP}}(\Gamma, \mathrm{Coind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}')$. But $[\Gamma : \mathrm{H}] < \infty$, so $\mathrm{Coind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}' = \mathrm{Ind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}'$. Therefore $H^1_{\mathcal{GP}}(\Gamma, \mathrm{Ind}^{\Gamma}_{\mathrm{H}}\mathfrak{g}')$ and hence $H^1_{\mathcal{GP}}(\Gamma, A)$ is nonzero. This completes the argument, because the non-vanishing of $H^1_{\mathcal{GP}}(\Gamma, A)$, by Theorem 3.4, implies that there exist Gorenstein derivations that are not principal. And the non-vanishing of $\widehat{H}^1(\Gamma, A)$ implies that there should exists derivations which are not Gorenstein.

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References

[1] Aldrich, S.T., Enochs, E.E. and Lopez Ramos, J.A., 2002. Derived functors of Hom relative to flat covers. Mathematische Nachrichten, 242(1), pp.17-26. https://doi.org/10.1007/s10013-014-0097-8.

[2] Asadollahi, J., Bahlekeh, A. and Salarian, S., 2009. On the hierarchy of cohomological dimensions of groups. Journal of Pure and Applied Algebra, 213(9), pp.1795-1803. https://doi.org/10.1016/j.jpaa.2009.01.11.

[3] Asadollahi, J., Bahlekeh, A., Hajizamani, A. and Salarian, S., 2011. On certain homological invariants of groups. Journal of Algebra, 335(1), pp.18-35. https://doi.org/10.1016/j.jalgebra.2011.03.018.

[4] Avramov, L.L. and Martsinkovsky, A., 2002. Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. Proceedings of the London Mathematical Society, 85(2), pp.393-440. https://doi.org/10.1112/500246115013527.

[5] Bahlekeh, A., Dembegioti, F. and Talelli, O., 2009. Gorenstein dimension and proper actions. Bulletin of the London Mathematical Society, 41(5), pp.859-871. https://doi.org/10.1112/blms/bdp063.

[6] Appelgate, H., Barr, M., Beck, J., Lawvere, F.W., Linton, F.E.J., Manes, E., Tierney, M. and Ulmer, F., 1969. Seminar on triples and categorical homology theory. Springer Berlin Heidelberg.

[7] Benkovič, D., 2009. Biderivations of triangular algebras. Linear algebra and its applications, 431(9), pp.1587-1602. https://doi.org/10.1016/laa.2009.05.029.

[8] Bennis, D. and Fahid, B., 2017. Derivations and the first cohomology group of trivial extension algebras. Mediterranean Journal of Mathematics, 14(4), p.150. Mediterr. https://doi.org/10.1007/s00009-017-0949-z

[9] Bican, L., El Bashir, R. and Enochs, E., 2001. All modules have flat covers. Bulletin of the London Mathematical Society, 33(4), pp.385-390. https://doi.org/10.1017/S0024609301008104.

[10] Bieri, R., 1976. Homological dimension of discrete groups. Queen Mary College Mathematics Notes.

[11] Bieri, R., 1979. Soluble groups with coherent group ring, in Homological Group Theory. London Math. Soc. Lecture Note Ser., 36, pp.235-240.

[12] Brown, K., 1982. Cohomology of Groups Springer Verlag. New York.

[13] Coelho, S.P. and Milies, C.P., 1993. Derivations of upper triangular matrix rings. Linear algebra and its applications, 187, pp.263-267. https://doi.org/10.1016/0024-3795(93)90141-A.

[14] Davidson, K.R., 1988. Pitman Research Notes in Mathematics Series.

[15] Enochs, E.E. and Jenda, O.M.G., 2000. Relative Homological Algebra, de Gruyter Exp. Mathesis, 30.

[16] Enochs, E.E., Jenda, O.M. and Torrecillas, B., Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan 10 (1993), no. 1, 1–9. MR 95a, 16004.

[17] Gruenberg, K.W., 2006. Cohomological topics in group theory (Vol. 143). Springer.

[18] Holm, H., 2004. Gorenstein derived functors. Proceedings of the American Mathematical Society, 132(7), pp.1913-1923. https://doi.org/10.1090/S0002-9939-04-07317-4.

[19] Holm, H., 2004. Gorenstein homological dimensions. Journal of pure and applied algebra, 189(1-3), pp.167-193. https://doi.org/10.1016/j.jpaa.2003.11.007.

[20] Jørgensen, P., 2007. Existence of Gorenstein projective resolutions and Tate cohomology. Journal of the European Mathematical Society, 9(1), pp.59-76. http://doi.org/10.4171/JEMS/72.

[21] Kropholler, P.H., 1993. On groups of type $(FP)_\infty$. Journal of pure and applied algebra, 90(1), pp.55-67. https://doi.org/10.1016/0022-4049(93)90136-H.

[22] MacLane, S., 2012. Homology. Springer Science & Business Media.

[23] Mitchell, B., 1965. Theory of categories (Vol. 17). Academic Press.

[24] Rotman, J.J. and Rotman, J.J., 2009. An introduction to homological algebra (Vol. 2). New York: Springer.

[25] Schmid, P., 1990. Frattinian p-groups. Geometriae Dedicata, 36(2), pp.359-364.

[26] Weibel, C.A., 1994. An introduction to homological algebra (No. 38). Cambridge university press.

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