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Journal of Advanced Mathematical Modeling
ISSN (print): 2251-8088, ISSN (on-line): 2645-6141
Vol. 15 No. 2 (2025), pp. 82-86.
DOI: 10.22055/jamm.2025.50224.2392

RING HOMOMORPHISMS BETWEEN MAXIMAL SUBRINGS OF REAL NUMBERS

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Communicated by: Mehrdad Namdari

Abstract. We prove that if S is a local radically closed subring of $\mathbb{R}$, then each ring homomorphism from S to $\mathbb{R}$ is the inclusion map. In particular, if S_1 and S_2 are maximal subrings of $\mathbb{R}$, then there exists a ring homomorphism $\phi : S_1 \to S_2$ if and only if $S_1 = S_2$ and in this case ϕ is the identity.

1. Introduction

The real numbers $\mathbb{R}$ is an important set in all of the subjects in mathematics and has an important role in the other sciences that basically depend on mathematics. Although its sophisticated construction is hard to come by, all scientific researchers, students and people use it without knowing its mysterious structure. In fact, the whole of mathematics somehow originated from the real numbers. There are many algebraic, topological and other results about the real numbers (see, for example [6]); however, we believe that many facts related to $\mathbb{R}$ are still unknown. For example, as we know, every set can be well-ordered by a relation, but what is a well-ordering relation on $\mathbb{R}$? This is a result whose existence is known, but no one has yet defined a well-ordering relation on $\mathbb{R}$. Nevertheless, having mentioned the mysterious character of $\mathbb{R}$ should not prevent us for finding new

MSC(2020): 00A09, 54C40.

Keywords: Ring homomorphism, local ring, integrally closed, maximal subring

Received: 20 July 2025, Accepted: 23 December 2025.

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facts about $\mathbb{R}$.

It is well known that the only ring homomorphism from $\mathbb{R}$ to itself, is the identity. Moreover, this property is inherited by maximal subrings of $\mathbb{R}$, too. In other words, if S is a maximal subrings of $\mathbb{R}$, then S is not a field (in fact S is a one-dimensional valuation domain) and the only ring homomorphism of S to itself, is the identity, see [1, Proposition 2.10 and Remark 2.11]. Moreover, in [2], the authors proved that whenever S and T are isomorphic maximal subrings of $\mathbb{R}$, then $S = T$. Before presenting our goal in this note, let us remind the reader some well known facts in algebra and set theory that are very similar to the results of this paper. The first one says that if M and N are simple left R-modules over arbitrary ring R, then $Hom_R(M, N) \neq 0$ if and only if M and N are isomorphic R-modules. The second one is a Theorem of Steinitz, see [4], that says if L and K are uncountable algebraically closed fields, then L and K are isomorphic fields if and only if L and K have same cardinality and character (i.e., $|K| = |L|$ and $Char(K) = Char(L)$). Finally, if α and β are two ordinals and $\phi : \alpha \to \beta$ is a bijective map which preserves the ordering, then $\alpha = \beta$.

Motivated by the previously mentioned results, in this note we prove that if S is a local radically closed subring of $\mathbb{R}$ and $\phi : S \to \mathbb{R}$ is a ring homomorphism, then ϕ is the inclusion map. In particular, we show that if S_1 and S_2 are maximal subrings of $\mathbb{R}$ and $\phi : S_1 \to S_2$ is a ring homomorphism, then $S_1 = S_2$ and ϕ is the identity. In other words, if S_1 and S_2 are maximal subrings of $\mathbb{R}$, and $Hom(S_1, S_2)$ be the set of all ring homomorphisms from S_1 to S_2, then we prove that $Hom(S_1, S_2) \neq \emptyset$ if and only if $S_1 = S_2$ and in particular in this case $Hom(S_1, S_2)$ only contains the identity map.

All rings in this note are commutative with $1 \neq 0$. All subrings, ring extensions and homomorphisms are unital. A proper subring S of a ring R is called a maximal subring if S is maximal with respect to inclusion in the set of all proper subrings of R. Not every ring possesses maximal subrings (for example, the algebraic closure of a finite field has no maximal subrings, see [1, Remark 1.13]). If R is a ring, then the prime subring of R is a subring of R which is generated by 1_R, i.e., $\{n.1_R \mid n \in \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the set of all integers. The characteristic of a ring R is denoted by $Char(R)$. If $R \subseteq T$ is a ring

extension, then $x \in T$ is called integral over R, whenever there exist a natural number n and $a_0, a_1, \ldots, a_{n-1}$ in R such that $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$. R is integrally closed in T, whenever $x \in T$ is integral over R, then $x \in R$. A subring R of a ring T is called radically closed if for each $x \in T$, whenever $x^2 \in R$, then we conclude that $x \in R$. Clearly each integrally closed subring R of a ring T is radically closed in T. A ring T is called local if it has a unique maximal ideal. For example, the localization of $\mathbb{Z}$ in any nonzero prime ideal (p), i.e., $\mathbb{Z}_{(p)} = \{\frac{m}{n} |\ m, n \in \mathbb{Z},\ n \neq 0\, n \notin p\mathbb{Z}\}$ is a local ring which is integrally closed in $\mathbb{Q}$. A subring V of a field K is called a valuation for K, if for each $x \in K \setminus \{0\}$, either $x \in V$ or $x^{-1} \in V$. It is not hard to see that each valuation domain is a local ring which is integrally closed in its quotient field. For example, $\mathbb{Z}_{(p)}$ is a valuation for $\mathbb{Q}$.

2. Main Result

Theorem 2.1. *Let S be a local radically closed subring of $\mathbb{R}$ and $\phi : S \to \mathbb{R}$ be a ring homomorphism, then ϕ is the inclusion map (i.e., for each $s \in S$ we have $\phi(s) = s$).*

Proof. Let M denote the unique maximal ideal of S. We prove the theorem by the following steps:

(1) First we claim that there exists a prime number p such that $\mathbb{Z}_{(p)} \subseteq S$. If $\mathbb{Q} \subseteq S$, then we are done. Hence assume that $\mathbb{Q} \nsubseteq S$. Therefore there exists a prime number p such that $1/p \notin S$. This immediately implies that $p \in M$ and therefore for each prime number $q \neq p$, we conclude that $q \notin M$ (note, there exist $a, b \in \mathbb{Z}$ such that $ap + bq = 1$ and therefore if $1/q \notin S$, then $q \in M$, i.e., $1 \in M$ which is absurd). Hence, $1/q \in S$ and thus $\mathbb{Z}_{(p)} \subseteq S$.

(2) Now we claim that $\phi(s) \geq 0$ for each $0 \leq s \in S$. To see this note that since $s \geq 0$, we conclude that $s = x^2$ for some $x \in \mathbb{R}$. Now since S is radically closed in $\mathbb{R}$, we deduce that $x \in S$ and therefore $\phi(s) = \phi(x)^2 \geq 0$. Thus for each $s_1 \leq s_2$ in S, we deduce that $\phi(s_1) \leq \phi(s_2)$.

(3) For each $q \in \mathbb{Z}_{(p)}$ we have $\phi(q) = q$. To see this note that $\phi(1) = 1$ and since ϕ is a ring homomorphism we obtain that $\phi(n) = n$ for each $n \in \mathbb{Z}$. Now if $q = m/n \in \mathbb{Z}_{(p)}$, then we have $m = \phi(m) = \phi(nq) = n\phi(q)$ and therefore $\phi(q) = m/n = q$.

(4) $Ker(\phi) = 0$. Let $\phi(s) = 0$, where $0 < s \in S$. Since $\mathbb{Z}_{(p)}$ is a dense subset of $\mathbb{R}$, we conclude that there exists $q \in \mathbb{Z}_{(p)}$ such that $0 < q < s$. Therefore by (1), (2)

and (3), we conclude that $0 = \phi(0) \leq q = \phi(q) \leq \phi(s) = 0$, which is absurd. Thus $Ker(\phi) = 0$.

Now it is clear that if $a < b$ are in S, then we have $\phi(a) < \phi(b)$. Now we prove that for each $s \in S$, we have $\phi(s) = s$. Assume that $\phi(s) \neq s$. We may suppose that $t = \phi(s)$ and $s < t$ (note, $\phi(-s) = -t$). Therefore there exists $q \in \mathbb{Z}_{(p)}$ such that $s < q < t$. This immediately implies that $t = \phi(s) < \phi(q) = q$ which is impossible. Thus ϕ is the inclusion map. □

Now we have the following immediate corollaries.

Corollary 2.2. *Let S be a local radically closed subring of $\mathbb{R}$ and T be a subring of $\mathbb{R}$. Then there a ring homomorphism $\phi : S \to T$ if and only if $S \subseteq T$ and in this case ϕ is the inclusion map.*

Corollary 2.3. *Let V be a local integrally closed subring $\mathbb{R}$ (in particular, if V is a valuation for $\mathbb{R}$) and $\phi : V \to \mathbb{R}$ be a ring homomorphism, then ϕ is the inclusion map.*

Corollary 2.4. *Let V be a valuation for $\mathbb{R}$ and T be a subring of $\mathbb{R}$. Then there exists a ring homomorphism $\phi : V \to T$ if and only if $T = V_P$, for some prime ideal P of V (and in this case ϕ is the inclusion).*

Proof. If $T = V_P$ for some prime ideal P of V, then clearly the map $phi(x) = \frac{x}{1}$ is a ring homomorphism. Conversely, assume that there exists a ring homomorphism ϕ from V to T, then by Corollary 2.3, ϕ is the inclusion map, i.e., V is a subring of T. Hence we are done by [5, Theorem 65]. □

Corollary 2.5. *Let S_1 and S_2 be maximal subrings of $\mathbb{R}$. If $\phi : S_1 \to S_2$ is a ring homomorphism, then $S_1 = S_2$ and ϕ is the identity.*

Proof. Assume that S_1 is a maximal subring of $\mathbb{R}$, then by [1, Remark 2.11], S_1 is not a field and therefore is a one-dimensional valuation domain. Hence by Corollary 2.3, we conclude that S_1 is a subring of S_2 and hence we are done by maximality of S_1. □

Acknowledgments

The author would like to thank the referee for his/her report. The author is grateful to the Research Council of Shahid Chamran University of Ahvaz (Ahvaz-Iran) for financial support (Grant Number: SCU.MM1404.721)

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