Shahid Chamran University of Ahvaz

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HOMOGENEOUS GENERALIZED RICCI SOLITONS ON THE LANCZOS UNIVERSE

AMIRHESAM ZAEIM*, GHASEM FOROZANI AND MARYAM DASHTI DARVISHZADEH

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Abstract. This work focuses on the study of Killing vector fields related to Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. After establishing the general properties of these symmetries, we concentrate on a specific subclass of FLRW models, known as the Lanczos universe. For this particular family of homogeneous spacetimes, we develop an algebraic description that captures their essential geometric structure. Using this algebraic approach, we also investigate the existence and properties of generalized Ricci solitons within the framework of Lanczos spacetimes.

1. Introduction

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric represents a fundamental solution to Einstein's field equations in general relativity, providing a mathematical description of a universe that is homogeneous (the same at every point) and isotropic (the same in all directions) [22, 14]. This metric forms the geometric foundation of the standard ΛCDM (Lambda Cold Dark Matter) cosmological model, which describes the large-scale structure, evolution, and dynamics of the universe [17, 20].

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*Corresponding author.

The FLRW spacetime in the standard local coordinates (t, r, θ, ϕ) is expressed by the line element :

$$ds^2 = c^2dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right), \tag{1.1}$$

where ds^2 measures the separation between events in spacetime, t is the cosmic time, and $a(t)$ is the scale factor, which determines how the universe expands or contracts over time. The curvature parameter k classifies the spatial geometry: $k = +1$ for a closed, spherical universe, $k = 0$ for a flat, Euclidean universe, and $k = -1$ for an open, hyperbolic universe [6].

While the FLRW metric describes a broad class of homogeneous and isotropic universes, certain specialized models within this framework possess distinctive geometric and physical features. One notable example is the *Lanczos universe*, which presents a unique solution to Einstein's equations within the FLRW family. Although less commonly investigated compared to other cosmological models, the Lanczos universe offers valuable insights into alternative geometric structures of spacetime and has been recognized for its distinctive mathematical structure [13, 21].

Originally introduced by Cornelius Lanczos [13], the Lanczos universe is a spatially homogeneous and isotropic cosmological model with positive spatial curvature ($k = +1$), characterized by the scale factor $a(t) = cb\cosh(t/b)$. It represents a closed de Sitter–type spacetime that undergoes a nonsingular bounce, smoothly connecting two asymptotically de Sitter phases. This model belongs to the family of nonsingular FLRW solutions introduced by Lanczos [13] and discussed in standard references [21, 11, 7].

Studies of the Lanczos universe have focused on its algebraic structure, dynamical behavior, and stability under various matter-energy distributions [9]. In particular, its non-trivial geometry provides fertile ground for investigating generalized Ricci solitons, symmetry properties, and the behavior of Killing vector fields within a cosmological setting [4].

Moreover, the Lanczos universe serves as a bridge between idealized, maximally symmetric models and more complex, realistic cosmological scenarios. Its rich geometric properties offer new perspectives on the mathematical classification of homogeneous spaces and contribute to the ongoing effort to understand the diversity of possible cosmic evolutions beyond standard isotropic models.

In this context, analyzing the Lanczos universe not only deepens our understanding of alternative solutions to Einstein's equations but also enhances our ability to model and interpret the universe's structural and dynamical complexities.

The structure of this work is organized as follows. In the next section, we review the fundamental concepts related to homogeneous spaces and provide the necessary background on generalized Ricci solitons. Section 3 is devoted to the investigation of Killing vector fields on FLRW spacetimes, where we derive the corresponding system of partial differential equations governing these symmetries. In Section 4, we turn our attention to the Lanczos universe, presenting a comprehensive analysis of its Killing vector fields and their algebraic properties. Finally, section 5 develops an algebraic approach for examining generalized Ricci solitons within the Lanczos spacetime, highlighting the interplay between geometric symmetries and soliton structures.

2. **Preliminaries**

In this section, we introduce foundational concepts required for the analysis developed in the subsequent parts of this work. Specifically, we provide a concise overview of (pseudo-)Riemannian homogeneous spaces and generalized Ricci solitons, both of which play a central role in understanding the geometric and algebraic structures considered later.

A *(pseudo-)Riemannian homogeneous space* (M, g) is a manifold equipped with a metric g, exhibiting the property that its isometry group $I(M)$ acts transitively on M. This implies that the geometry of the space is uniform—any two points are indistinguishable under isometries. Such spaces can be identified with coset spaces $M \simeq G/H$, where $G = I(M)$ is the full isometry group, and H is the isotropy subgroup fixing a chosen base point.

The infinitesimal structure of these spaces is encoded via the Lie algebra $\mathfrak{g}$ of G, and the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ corresponds to $\mathfrak{Lie}(H)$, the Lie algebra of the isotropy sublagebra, and $\mathfrak{m}$ identifies with the tangent space at the origin. The isotropy representation φ is described by:

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m}), \quad \varphi(x)(y) = [x, y]_{\mathfrak{m}}, \quad \forall\, x \in \mathfrak{g},\, y \in \mathfrak{m}, \tag{2.1}$$

ensuring that the metric g remains invariant under the action of H:

$$^{t}\varphi(x) \cdot g + g \cdot \varphi(x) = 0, \quad \forall\, x \in \mathfrak{h}. \tag{2.2}$$

The geometry is further described by the canonical connection Λ:

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + \nu(x, y), \tag{2.3}$$

where ν is a symmetric bilinear map satisfying the compatibility condition:

$$2g\big(\nu(x, y), z_{\mathfrak{m}}\big) = g\big(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}\big) + g\big(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}\big),$$

for all $x, y, z \in \mathfrak{g}$.

The curvature of the space is captured algebraically via:

$$R : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}), \quad R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y]), \tag{2.4}$$

from which one can derive the Ricci tensor ϱ and scalar curvature τ through:

$$\varrho(x, y) = \operatorname{tr}\big(z \mapsto R(z, x)y\big), \quad \tau = \operatorname{tr}_g(\varrho).$$

This algebraic characterization of homogeneous spaces provides a powerful motivation for analyzing different geometric models in general relativity. For example, homogeneous Siklos spacetimes were studied through their Lie algebra structures in [23], while the broader role of algebraic methods in understanding non-reductive homogeneous pseudo-Riemannian manifolds was established in the work of Fels and Renner [8]. Similar techniques have also been applied to the classification and geometric study of homogeneous Gödel-type spacetimes [2].

Beyond these classical structures, Ricci solitons provide a natural generalization of Einstein manifolds. A *Ricci soliton* is a (pseudo-)Riemannian manifold (M, g) equipped with a vector field V satisfying:

$$\mathcal{L}_V g + \varrho = \lambda g, \tag{2.5}$$

where $\mathcal{L}_V$ denotes the Lie derivative along V, and λ is a real constant. Depending on the sign of λ, Ricci solitons are classified as shrinking ($\lambda > 0$), steady ($\lambda = 0$), or expanding ($\lambda < 0$). Ricci solitons model self-similar solutions to the Ricci flow and often arise in the study of singularities or long-term geometric evolution.

A more general framework is provided by the *generalized Ricci soliton* equation (see [15]):

$$\mathcal{L}_V g + 2\eta V^\flat \odot V^\flat - 2\beta\varrho = 2\lambda g, \tag{2.6}$$

where $\eta, \beta, \lambda \in \mathbb{R}$ are scalar parameters, $V^\flat$ is the metric dual 1-form associated to V, and $\odot$ denotes the symmetric product. This extension introduces additional flexibility, capturing a wider class of geometric flows, soliton-like structures, and possible physical interpretations [4].

Several notable special cases of (2.6) include:

- **Classical Ricci Solitons:** Recovered when $\eta = 0$ and $\beta = -1$, originally introduced in the study of self-similar solutions to the Ricci flow [10].
- **Gradient Ricci Solitons:** If $V = \nabla f$ for some smooth potential f, $\eta = 0$, and $\beta = -1$, yielding:

$$\nabla^2 f + \varrho = \lambda g,$$

  which are of fundamental importance in understanding singularities of Ricci flow [5].
- **Trivial Solutions (Killing Fields):** If $\eta = \beta = \lambda = 0$, the equation reduces to $\mathcal{L}_V g = 0$, identifying V as a Killing vector field [16].
- **Curvature-Modified Solitons:** When $\beta \neq -1$, the Ricci tensor enters with a rescaled contribution, modifying the flow's geometric response, allowing for diverse extensions in geometric analysis [4].
- **Einstein–Weyl structures:** For the parameter choice $\eta = 1$ and $\beta = -\frac{1}{n-2}$ (with $n > 2$), the generalized Ricci soliton equation (2.6) reduces to the *Einstein–Weyl equation* in conformal geometry. In this case, the vector field V represents the Weyl potential of a torsion–free Weyl connection whose symmetric, trace–free Ricci part vanishes [1].
- **Metric projective structures with skew–symmetric Ricci representative:** Setting $\eta = 1$, $\beta = -\frac{1}{n-1}$, and $\lambda = 0$, equation (2.6) describes the condition that a metric projective structure admits a *skew–symmetric Ricci tensor representative* within its projective class. In dimension two, this specialization corresponds to the projective Einstein–Weyl system [19].
- **Vacuum near–horizon geometry equation:** For $\eta = 1$ and $\beta = \frac{1}{2}$, equation (2.6) reproduces the vacuum near–horizon geometry equation from general relativity, with λ interpreted as the cosmological constant [15].

- **Einstein Spaces as Trivial Solitons:** For $V = 0$, (2.6) simplifies to:

$$\varrho = -\frac{\lambda}{\beta} g,$$

recovering the Einstein condition, provided $\beta \neq 0$ [21].

Critical Observations:

- The balance between curvature contributions and nonlinear vector–driven deformations in the generalized Ricci soliton equation is governed by the parameters η and β. They determine the behavior of the soliton under the combined influence of the Ricci tensor and the vector term $V^\flat \odot V^\flat$. Certain parameter choices specialize (2.7) to well–known geometric types (e.g., Einstein metrics when $V = 0$ and the classical Ricci–soliton case), with illustrative Lorentzian examples in Gödel-type analyses [3] and in Lorentzian Walker settings [18]. This parameter flexibility interpolates between curvature-dominated and vector-dominated regimes.
- The quadratic term $V^\flat \odot V^\flat$; for $\eta > 0$ its influence grows with $\|V\|$, and in Lorentzian settings the causal structure can amplify its nonlinear impact. Such terms can lead to steady or expanding soliton configurations with anisotropic features; see homogeneous Siklos examples [23]. An analogous vector–curvature coupling appears for generalized Ricci-type solitons on certain CR-manifolds [12].
- The parameter β tunes the weight of the Ricci tensor in (2.7), adjusting the balance between curvature and the vector term. In the *classical Ricci–soliton* specialization ($\eta = 0$, $\beta = -1$), the standard shrinking/steady/expanding classification is determined by the sign of λ; see Lorentzian Gödel-type models [3]. In the fully generalized setting, varying β rescales the Ricci contribution but the shrinking/steady/expanding nature remains governed by λ.
- Generalized Ricci solitons encompass both self-similar evolutions and steady-state configurations: they capture equilibria where curvature and vector effects balance, as well as metric self-similarity under diffeomorphisms and scaling. Examples exhibiting steady or expanding regimes in homogeneous Lorentzian settings include the Siklos family [23]; see also related phenomena for generalized Ricci-type solitons on CR-manifolds [12].

In analogy with the classical case, a generalized Ricci soliton can be viewed as a self-similar or steady solution of a generalized Ricci-type flow whose right-hand side reproduces

the terms of the soliton equation:

$$\frac{\partial g}{\partial t} = 2\beta\,\varrho - 2\eta\,V^\flat \odot V^\flat,$$

up to pull-back by the diffeomorphisms generated by V. For $\eta = 0$ and $\beta = -1$, one recovers Hamilton's Ricci flow $\partial_t g = -2\varrho$, so the generalized soliton equation represents its natural extension incorporating nonlinear vector effects. In this interpretation, generalized Ricci solitons describe equilibrium or self-similar geometries of the above flow, with λ distinguishing shrinking, steady, and expanding regimes.

All things considered, homogeneous spaces and generalized Ricci solitons are fundamental building blocks for investigating symmetry, curvature, and geometric evolution, especially in cosmological models where both isotropic and anisotropic structures naturally occur.

This summary follows Table 1 of [15], showing that many familiar curvature–vector coupling equations appear as particular parameter regimes of (2.6).

TABLE 1. Representative specializations of the generalized Ricci soliton equation (2.6).

Geometric type	$\boldsymbol{\eta}$	$\boldsymbol{\beta}$	$\boldsymbol{\lambda}$
Killing's equation	0	0	0
Homotheties	0	0	arbitrary
Classical (gradient) Ricci solitons	0	-1	arbitrary
Curvature-Modified Solitons	*	$\neq -1$	arbitrary
Einstein–Weyl structures ($n > 2$)	1	$-\frac{1}{n-2}$	arbitrary
Metric projective structures with skew–symmetric Ricci tensor	1	$-\frac{1}{n-1}$	0
Vacuum near–horizon geometry equations	1	$\frac{1}{2}$	arbitrary

3. **Killing vector fields of FLRW spacetimes**

The FLRW metric, expressed in Equation (1.1), serves as the foundation for our analysis. For simplicity in tensorial computations, we set the coordinates as $(t, r, \theta, \phi) = (x_1, x_2, x_3, x_4)$. Consider an arbitrary vector field

$$X = X^1(x_1, x_2, x_3, x_4)\partial_1 + X^2(x_1, x_2, x_3, x_4)\partial_2 + X^3(x_1, x_2, x_3, x_4)\partial_3 + X^4(x_1, x_2, x_3, x_4)\partial_4,$$

where each component X^i is a smooth function of the variable (x_1, x_2, x_3, x_4) and $\partial_i := \frac{\partial}{\partial x_i}$ are coordinate vector fields. The non-vanishing Christoffel symbols associated with the Friedmann-Lemaître-Robertson-Walker (FLRW) metric can be expressed in the following explicit form:

$$\begin{aligned}
&\Gamma^2_{21} = \Gamma^3_{31} = \Gamma^4_{41} = \tfrac{a'(x_1)}{a(x_1)}, && \Gamma^1_{22} = -\tfrac{a(x_1)a'(x_1)}{c^2(kx_2^2-1)},\\
&\Gamma^2_{22} = -\tfrac{kx_2}{kx_2^2-1}, && \Gamma^3_{32} = \tfrac{1}{x_2},\\
&\Gamma^4_{42} = \tfrac{1}{x_2}, && \Gamma^1_{33} = \tfrac{x_2^2 a(x_1)a'(x_1)}{c^2},\\
&\Gamma^2_{33} = x_2(kx_2^2-1), && \Gamma^4_{43} = \cot(x_3),\\
&\Gamma^1_{44} = \tfrac{x_2^2\sin^2(x_3)a(x_1)a'(x_1)}{c^2}, && \Gamma^2_{44} = x_2\sin^2(x_3)(kx_2^2-1),\\
&\Gamma^3_{44} = -\sin(x_3)\cos(x_3).
\end{aligned}$$

We compute the Lie derivative $\mathcal{L}_X g$ of the metric tensor g along the prescribed vector field X. The Lie derivative of the metric tensor can be formulated in terms of covariant derivatives as:

$$(\mathcal{L}_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

The Killing condition enforces the nullity of these derivatives, leading to the subsequent constraints:

(3.1)
$$\begin{cases} \partial_1 X^1 = 0, \\ \sin^2(x_3)\partial_3 X^4 + \partial_4 X^3 = 0, \\ x_2^2 \partial_2 X^3 + \frac{1}{1-kx_2^2}\partial_3 X^2 = 0, \\ c^2 \partial_3 X^1 - a(x_1)^2 x_2^2 \partial_1 X^3 = 0, \\ a(x_1)^2 \partial_1 X^2 + c^2(kx_2^2 - 1)\partial_2 X^1 = 0, \\ c^2 \partial_4 X^1 - a(x_1)^2 x_2^2 \sin^2(x_3)\partial_1 X^4 = 0, \\ a(x_1)^2 x_2^2 \sin^2(x_3)\partial_2 X^4 - \frac{1}{kx_2^2-1}\partial_4 X^2 = 0, \\ a(x_1)x_2 \partial_3 X^3 + a'(x_1)x_2 X^1 + a(x_1)X^2 = 0, \\ a(x_1)\partial_2 X^2 + a'(x_1)X^1 + \frac{a(x_1)kx_2}{1-kx_2^2}X^2 = 0, \\ a(x_1)x_2 \sin(x_3)\partial_4 X^4 + a'(x_1)x_2 \sin(x_3)X^1 + a(x_1)\sin(x_3)X^2 + a(x_1)x_2\cos(x_3)X^3 = 0. \end{cases}$$

4. **Analysis of Killing vector fields on the Lanczos universe**

The *Lanczos universe* is a notable exact solution to Einstein's field equations, introduced by Cornelius Lanczos in 1924 [13]. It is a special case of the Friedmann–Lemaître–Robertson–Walker (FLRW) class of cosmological models, characterized by homogeneity, isotropy, and positive spatial curvature. Unlike standard big bang models, the Lanczos universe avoids an initial singularity and instead describes a bouncing cosmology with exponential behavior at early and late times.

According to the metric of the equation (1.1), the scale factor in this model is:

$$a(t) = cb\cosh\left(\frac{t}{b}\right),$$

which is symmetric under time reversal and possesses a minimum value at $t = 0$, i.e., $a_{\min} = cb$. This implies that the universe undergoes a nonsingular bounce, contracting to a finite minimum size before re-expanding.

The line element of the Lanczos universe in comoving coordinates (t, r, θ, ϕ) is given by:

$$ds^2 = c^2dt^2 - (cb)^2 \cosh^2\left(\frac{t}{b}\right)\left(\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right),$$

where c is the speed of light, b is a constant with dimensions of time. This metric represents a closed universe ($k = +1$) filled with a vacuum energy component equivalent to a positive cosmological constant.

The Lanczos universe provides a valuable exact solution for exploring nonsingular cosmological models. Its exponential expansion at large $|t|$ mimics de Sitter behavior, making it relevant in studies of early-universe inflation and quantum cosmology [11, 7].

By setting the scale factor of the Lanczos universe and $k = 1$ in the system of equations for Killing vector fields of the equation (3.1), and by standard calculations, one gets the following family of ten Killing vector fields:

$$
\begin{aligned}
K_1 &= \sqrt{x_2^2-1}\partial_1 - \frac{x_2}{b}\tanh\left(\frac{x_1}{b}\right)\sqrt{x_2^2-1}\partial_2,\\
K_2 &= x_2\cos(x_3)\,\partial_1 + \frac{(1-x_2^2)}{b}\cos(x_3)\tanh\left(\frac{x_1}{b}\right)\partial_2 - \frac{1}{bx_2}\sin(x_3)\tanh\left(\frac{x_1}{b}\right)\,\partial_3,\\
K_3 &= x_2\sin(x_3)\sin(x_4)\partial_1 + \frac{(1-x_2^2)}{b}\sin(x_3)\sin(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_2\\
&\quad + \frac{1}{bx_2}\cos(x_3)\sin(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_3 + \frac{1}{bx_2\sin(x_3)}\cos(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_4,\\
K_4 &= x_2\sin(x_3)\cos(x_4)\partial_1 + \frac{(1-x_2^2)}{b}\sin(x_3)\cos(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_2\\
&\quad + \frac{1}{bx_2}\cos(x_3)\cos(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_3 - \frac{1}{bx_2\sin(x_3)}\sin(x_4)\tanh\left(\frac{x_1}{b}\right)\partial_4,\\
K_5 &= \cos(x_3)\sqrt{x_2^2-1}\partial_2 - \frac{\sin(x_3)}{x_2}\sqrt{x_2^2-1}\partial_3,\\
K_6 &= \sin(x_3)\sin(x_4)\sqrt{x_2^2-1}\partial_2 + \frac{\cos(x_3)\sin(x_4)}{x_2}\sqrt{x_2^2-1}\partial_3 + \frac{\cos(x_4)}{x_2\sin(x_3)}\sqrt{x_2^2-1}\partial_4,\\
K_7 &= \sin(x_3)\cos(x_4)\sqrt{x_2^2-1}\partial_2 + \frac{\cos(x_3)\cos(x_4)}{x_2}\sqrt{x_2^2-1}\partial_3 - \frac{\sin(x_4)}{x_2\sin(x_3)}\sqrt{x_2^2-1}\partial_4,\\
K_8 &= \partial_4,\\
K_9 &= -\cos(x_4)\partial_3 + \cot(x_3)\sin(x_4)\partial_4,\\
K_{10} &= \sin(x_4)\partial_3 + \cot(x_3)\cos(x_4)\partial_4.
\end{aligned}
$$

Now, the isotropy subalgebra in the base point $(0, \frac{\sqrt{2}}{2}, \frac{\pi}{2}, 0)$ is generated by the following vectors

$$
\begin{cases}
e_1 &= K_1 - iK_4,\\
e_2 &= K_2,\\
e_3 &= K_3,\\
e_4 &= K_5 - iK_9,\\
e_5 &= K_6 - iK_8,\\
e_6 &= K_{10},
\end{cases}
$$

where i represents the imaginary unit. By setting $u_1 = K_1, u_2 = K_5, u_3 = K_6$ and $u_4 = K_7$, the Lie algebra in the basis $\{e_1, e_2, e_3, e_4, e_5, e_6, u_1, u_2, u_3, u_4\}$ is determined as

	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3	u_4
e_1	0	$\frac{e_4}{b^2}$	$\frac{e_5}{b^2}$	$-2e_2$	$-2e_3$	0	$\frac{iu_4}{b^2}$	$-e_2$	$-e_3$	$2iu_1 - ie_1$
e_2	$\frac{-e_4}{b^2}$	0	$\frac{e_6}{b^2}$	$-e_1$	0	e_3	$\frac{-u_2}{b^2}$	$-u_1$	0	0
e_3	$\frac{-e_5}{b^2}$	$\frac{-e_6}{b^2}$	0	0	$-e_1$	$-e_2$	$\frac{-u_3}{b^2}$	0	$-u_1$	0
e_4	$2e_2$	e_1	0	0	$2e_6$	e_5	e_2	$-iu_4$	e_6	$2iu_2 - ie_4$
e_5	$2e_3$	0	e_1	$-2e_6$	0	$-e_4$	e_3	$-e_6$	$-iu_4$	$2iu_3 - ie_5$
e_6	0	$-e_3$	e_2	$-e_5$	e_4	0	0	$-u_3$	u_2	0
u_1	$\frac{-iu_4}{b^2}$	$\frac{u_2}{b^2}$	$\frac{u_3}{b^2}$	$-e_2$	$-e_3$	0	0	$-e_2$	$-e_3$	$iu_1 - ie_1$
u_2	e_2	u_1	0	iu_4	e_6	u_3	e_2	0	e_6	$iu_2 - ie_4$
u_3	e_3	0	u_1	$-e_6$	iu_4	$-u_2$	e_3	$-e_6$	0	$iu_3 - ie_5$
u_4	$-2iu_1 + ie_1$	0	0	$-2iu_2 + ie_4$	$-2iu_3 + ie_5$	0	$-iu_1 + ie_1$	$-iu_2 + ie_4$	$-iu_3 + ie_5$	0

Now, using the relation (2.1), isotropy representations corresponding to the generators $h_1, \cdots, h_6$ are calculated as

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 2i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{b^2} & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{1}{b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad H_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad H_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, equation (2.2) yields that an arbitrary symmetric matrix g is an invariant metric if and only if ${}^tH_s g + gH_s = 0$ for $s = 1, \cdots, 6$. Thus, the invariant metric is directly

calculated as

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -ab^2 & 0 & 0 \\ 0 & 0 & -ab^2 & 0 \\ 0 & 0 & 0 & -2ab^2 \end{pmatrix}, \tag{4.1}$$

where $a \neq 0$ is an arbitrary real constant.

5. **Generalized Ricci solitons on the Lanczos universe**

We now direct our attention to the Ricci soliton equation (2.5). To this end, it is necessary to compute the connection, curvature, and related geometric quantities. All computations are carried out in the complexified tangent algebra, and the reality condition is imposed at the end by restricting to real-valued tensors. According to equation (2.3), setting $\Lambda_i := \Lambda(u_i)$, the components of the Levi-Civita connection with respect to the basis $\{u_1, \cdots, u_4\}$ of $\mathfrak{m}$ and the dual basis $\{\omega^1, \cdots, \omega^4\}$ are fully determined as follows

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\mathrm{i}}{2b^2} & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\dfrac{\mathrm{i}}{2} & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\dfrac{\mathrm{i}}{2} & 0 \end{pmatrix}, \qquad \Lambda_4 = 0.$$

Now, matrices of curvature operators are calculated by the equation (2.4) as

$$R_{12} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2b^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{14} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2b^2} & 0 & 0 & 0 \end{pmatrix}, \quad R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad R_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

Finally, the Ricci tensor is obtained as follows

$$\varrho = \begin{pmatrix} \frac{3}{2b^2} & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Let $V = v_1u_1 + v_2u_2 + v_3u_3 + v_4u_4$ be an arbitrary vector field on the homogeneous Lanczos space $(G/H, g)$. Note that the components v_i of the potential vector field V are constant with respect to the basis $\{u_1, \ldots, u_4\}$ of the factor space $\mathfrak{m}$. Using the explicit form of the Levi-Civita connection, the Lie derivative of the metric g with respect to V is given by:

$$\mathcal{L}_V g = \begin{pmatrix} 2i\,v_4\,a & 0 & 0 & -i\,v_1\,a \\ 0 & -2i\,v_4\,a\,b^2 & 0 & i\,v_2\,a\,b^2 \\ 0 & 0 & -2i\,v_4\,a\,b^2 & i\,v_3\,a\,b^2 \\ -i\,v_1\,a & i\,v_2\,a\,b^2 & i\,v_3\,a\,b^2 & 0 \end{pmatrix}$$

According to the Ricci soliton equation (2.5), the following system of equations is deduced:

$$\begin{cases} v_1 a = 0, \\ v_2 ab = 0, \\ v_3 ab = 0, \\ 3 - 2\,\eta\, a\, b^2 = 0, \\ 2i\, v_4\, a + \dfrac{3}{2b^2} - \eta\, a = 0, \\ 2iv_4 ab^2 + \dfrac{3}{2} - \eta\, a\, b^2 = 0. \end{cases}$$

This system implies the vector field V must vanish, indicating that no non-trivial Ricci solitons exist on the homogeneous Lanczos space.

Remark 5.1. The homogeneous Lanczos space $(G/H, g)$ does not admit any non-trivial Ricci solitons.

We now turn to generalized Ricci solitons. The symmetric tensor $V^\flat \odot V^\flat$ reads:

$$V^\flat \odot V^\flat = a^2 \begin{pmatrix} v_1^2 & -v_1\, v_2\, b^2 & -v_1\, v_3\, b^2 & -2v_1\, v_4\, b^2 \\ -v_1\, v_2\, b^2 & v_2^2\, b^4 & v_2\, v_3\, b^4 & 2v_2\, v_4\, b^4 \\ -v_1\, v_3\, b^2 & v_2\, v_3\, b^4 & v_3^2\, b^4 & 2v_3\, v_4\, b^4 \\ -2v_1\, v_4\, b^2 & 2v_2\, v_4\, b^4 & 2v_3\, v_4\, b^4 & 4v_4^2\, b^4 \end{pmatrix}$$

Then, using the equation (2.6), we derive the following necessary and sufficient conditions:

$$\begin{cases} 2\eta v_1 v_2 a^2 b^2 = 0, \\ 2\eta v_1 v_3 a^2 b^2 = 0, \\ 2\eta v_2 v_3 a^2 b^4 = 0, \\ i v_1 a + 4\eta v_1 a^2 v_4 b^2 = 0, \\ i\, v_2\, a\, b^2 + 4\,\eta\, v_2\, a^2\, b^4\, v_4 = 0, \\ i\, v_3\, a\, b^2 + 4\,\eta\, v_3\, a^2\, b^4\, v_4 = 0, \\ 8\,\eta\, v_4^2\, a^2\, b^4 + 6\,\beta + 4\,\lambda\, a\, b^2 = 0, \\ 2i\, v_4\, a + 2\,\eta\, v_1^2\, a^2 - \dfrac{3\,\beta}{b^2} - 2\,\lambda\, a = 0, \\ 2i v_4 a b^2 - 2\eta v_2^2 a^2 b^4 - 3\beta - 2\lambda a b^2 = 0, \\ 2i v_4 a b^2 - 2\eta v_3^2 a^2 b^4 - 3\beta - 2\lambda a b^2 = 0. \end{cases}$$

We assume $\eta \neq 0$ to focus on non-trivial generalized Ricci solitons. Since both g and ϱ are diagonal in this invariant frame, and taking into account the contribution of $\mathcal{L}_V g$, the off-diagonal components $((i,j) = (1,2),(1,3),(2,3))$ of the generalized Ricci soliton equation reduce to

$$2\eta \left(V^\flat \odot V^\flat\right)_{ij} = 0, \qquad (i,j) = (1,2),(1,3),(2,3).$$

This condition immediately implies that at least two of the coefficients among v_1, v_2, v_3 must vanish. Now, standard calculations yield that assuming non-vanishing any one of these coefficients would give a contradiction. For instance, when $v_1 = v_2 = 0$, the ninth equation yields $v_4 = -\frac{i(3\beta+2\lambda a b^2)}{2ab^2}$, and consequently, the tenth equation becomes $2\eta v_3^2 a^2 b^4 = 0$, implying that $v_3 = 0$. Thus, we should have $v_1 = v_2 = v_3 = 0$. Then, seventh and eighth equations give

$$v_4 = -\frac{i}{2\eta a b^2}, \quad \beta = \frac{1 - 2\lambda a b^2 \eta}{3\eta}.$$

Any soliton vector V aligned with u_4 is spacelike since the vector u_1 is timelike and u_4 is spacelike with respect to the invariant metric (4.1) (assuming $a < 0$). The isotropic structure of the geometry is encoded by such invariant vector fields, which are crucial for

describing the symmetry of homogeneous spacetimes and identifying the direction along which the metric admits soliton-type deformations.

We summarize the above results in the following theorem.

Theorem 5.2. *A homogeneous Lanczos space $(G/H, g)$ is a generalized Ricci soliton if and only if the vector field V satisfies*

$$V = -\frac{i}{2\eta ab^2}u_4,$$

and $\beta = \frac{1-2\lambda ab^2\eta}{3\eta}$. Moreover, there are no generalized Ricci solitons in the field of Real numbers.

6. **Conclusion**

In this study, we examined into the geometric properties of the Lanczos universe, a notable exact solution within the Friedmann-Lemaître-Robertson-Walker (FLRW) class of cosmological models. Our primary objective was to investigate the existence and characteristics of generalized Ricci solitons within this specific spacetime. We began by thoroughly analyzing the Killing vector fields of general FLRW spacetimes, establishing a system of partial differential equations that govern these fundamental symmetries. This analysis laid the groundwork for our specialized examination of the Lanczos universe.

For the Lanczos universe, characterized by its scale factor $a(t) = cb\cosh\left(\frac{t}{b}\right)$ and positive spatial curvature ($k = +1$), we identified a ten-dimensional family of Killing vector fields. These symmetries are consistent with the known properties of maximally symmetric spacetimes like de Sitter space, which the Lanczos universe approaches asymptotically for large $|t|$ [11, 7]. Our algebraic description of these Killing vector fields and the subsequent derivation of the invariant metric (Equation (4.1)) provide a robust framework for further geometric investigations. The metric obtained, $g = \mathrm{diag}(a, -ab^2, -ab^2, -2ab^2)$, with $a \neq 0$, captures the structure of the homogeneous space.

A key finding of our investigation concerns the existence of Ricci solitons on the homogeneous Lanczos space. Our analysis, based on Equation (2.5), revealed that no non-trivial Ricci solitons exist in this context. This result is significant as it indicates that the homogeneous Lanczos space does not admit self-similar solutions to the Ricci flow that are driven solely by the Ricci curvature and a Killing vector field, a property distinguishing it from other types of manifolds that may support such solitons [4]. This absence suggests a

certain rigidity or specific evolutionary behavior of this spacetime under Ricci flow, implying that any geometric evolution would not reduce to a simple scaling and diffeomorphism generated by a non-trivial vector field.

Furthermore, we extended our analysis to generalized Ricci solitons, governed by Equation (2.6), which incorporates additional parameters η and β. Our findings demonstrate that a homogeneous Lanczos space $(G/H, g)$ can be a generalized Ricci soliton if and only if the vector field V is of the form $V = -\frac{i}{2\eta ab^2}\partial_4$, with the condition $\beta = \frac{1-2\lambda ab^2\eta}{3\eta}$. Crucially, the presence of the imaginary unit i in the components of the vector field V and the derived relation for β implies that **there are no generalized Ricci solitons in the field of real numbers** for the homogeneous Lanczos space. This novel result highlights a fundamental limitation on the types of geometric flows that this spacetime can support under the generalized Ricci soliton framework, at least within a real-valued context.

Comparing these results with established facts, the nonsingular bouncing nature of the Lanczos universe [13, 21] typically implies a more complex dynamical structure than standard singular cosmological models. Our findings regarding the absence of real-valued (generalized) Ricci solitons suggest that the evolution of the Lanczos universe, as described by these flow equations, does not simplify to the self-similar forms usually associated with such solitons in real pseudo-Riemannian geometries. This aligns with the understanding that the Lanczos universe, while homogeneous and isotropic in its spatial sections, possesses unique characteristics that set it apart from typical cosmological solutions, particularly in how its geometry responds to deformation under generalized Ricci flow.

Numerous pseudo-Riemannian (specially Lorentzian) manifolds have been the subject of in-depth research on generalized Ricci solitons. As an example, Calvaruso identified both Riemannian and Lorentzian models for classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups, as well as three-dimensional homogeneous generalized Ricci solitons. [3]. Additional examples were found on three-dimensional Lorentzian Walker manifolds [18] and homogeneous Siklos spacetimes, where explicit families of generalized Ricci solitons were constructed [23].

The homogeneous *Lanczos universe* considered in this work, in contrast to these examples, reveals a higher degree of geometric rigidity and does not support any real generalized

Ricci soliton solution. This highlights how unique the Lanczos metric is for homogeneous Lorentzian spacetimes and how resistant it is to significant deformations under generalized Ricci flow.

Appendix

The Maple software code used to compute Killing vector fields is as follows.

```
with(DifferentialGeometry):
with(Tensor):
with(DifferentialGeometry:-Library):
# 1) Manifold and coordinates.
DGsetup([t, r, theta, phi], M):
# 2) Scale factor (Lanczos / closed de Sitter-type)
# a(t) = c b cosh(t/b), with b¿0, c¿0, k=1
assume(b¿0, c¿0):
a := c*b*cosh(t/b):k:=1:
# 3) Metric g on M (signature + - - -)
# ds^2 = c^2dt^2 − a(t)^2[dr^2/(1 − kr^2) + r^2dθ^2 + r^2sin^2(θ)dφ^2]
g := evalDG(c^2* dt&t dt-a^2 *((1/(1-k*r^2)) * dr&t dr+r^2 * dtheta&t dtheta+r^2
* sin(theta)^2 * dphi &t dphi));
# 4) Register the metric with DifferentialGeometry (so Christoffels, curvatures, etc. us
it)
SetupMetric(g):
# 5) Compute the Lie algebra of Killing vector fields
K := KillingVectors(g):
nops(K), K;
```

References

[1] Calderbank, D.M.J. and Pedersen, H., 1999. Einstein-Weyl Geometry In: Surveys in differential geometry: Essays on Einstein manifolds. *Surveys in Differential Geometry*, 6, pp. 387–423.

[2] Calvaruso, G., 2018. The Ricci soliton equation and the structure of homogeneous Gödel-type spacetimes. *J. Math. Anal. Appl.*, 465(2), 1112–1133. doi:10.1016/j.jmaa.2018.05.051

[3] Calvaruso, G., 2017. Three-Dimensional Homogeneous Generalized Ricci Solitons. *Mediterranean Journal of Mathematics*, 14(5)(2017), Article 216. doi:10.1007/s00009-017-1019-2

[4] Calviño-Louzao, E., Garcia-Rio, E. and Vazquez-Lorenzo R., 2017. *Ricci Solitons: Aspects of differential geometry III*. Springer Cham. doi:10.1007/978-3-031-02410-8.

[5] Cao, H.-D., 2010. Recent progress on Ricci solitons. *Recent Advances in Geometric Analysis, Adv. Lect. Math. (ALM) 11*, Int. Press, Somerville, MA, pp. 1–38. doi:10.48550/arXiv.0908.2006

[6] Dodelson, S. and Fabian, S., 2020. *Modern Cosmology*. London: Academic Press. doi:10.1016/C2017-0-01943-2

[7] Ellis, G.F.R., Maartens, R. and MacCallum, M.A.H., 2012. *Relativistic Cosmology*. Cambridge University Press. www.cambridge.org/9780521381154

[8] Fels, M. E. and Renner, A. G., 2006. Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four. *Canadian Journal of Mathematics*, 58(2), pp. 282–311. doi:10.4153/CJM-2006-012-1

[9] Griffiths, J. B. and Podolský, J., 2009. *Exact Space-Times in Einstein's General Relativity*. Cambridge University Press. doi:10.1017/CBO9780511635397

[10] Hamilton, R. S., 1988. The Ricci flow on surfaces. *Contemporary Mathematics*, 71, 237–261. doi:10.1090/conm/071/954419

[11] Hawking, S.W. and Ellis, G.F.R., 1973. *The Large Scale Structure of Space-Time*. Cambridge University Press. doi:10.1017/CBO9780511524646

[12] Kumara H. A., Naik D. M. and Venkatesha V., 2022. Geometry of generalized Ricci-type solitons on a class of Riemannian manifolds. *J. Geom. Phys.* 175, 104507. doi:10.1016/j.geomphys.2022.104506

[13] Lanczos, C., 1924. Über eine stationäre Kosmologie im Sinne der Einsteinischen Gravitationstheorie. *Zeitschrift für Physik*, 21, pp. 73–110.

[14] Mukhanov, V., 2005. *Physical Foundations of Cosmology*. Cambridge University Press. doi:10.1017/CBO9780511790553

[15] Nurowski, P. and Randall, M., 2016. Generalised Ricci Solitons. *J. Geometric Analysis*, 26, pp. 1280–1345. doi:10.1007/s12220-015-9592-8

[16] O'Neill, B., 1983. *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press. doi:10.1112/blms/16.3.328

[17] Peebles, P. J. E., 1993. *Principles of Physical Cosmology*. Princeton University Press. doi:10.1515/9780691206721

[18] Pirhadi, V., Fasihi-Ramandi, G. and Azami, S., 2023. Generalized Ricci Solitons on Three-Dimensional Lorentzian Walker Manifolds. *J. Nonlinear Mathematical Physics*, 30, pp. 1409–1423. doi:10.1007/s44198-023-00134-4

[19] Randall, M., 2014. Local obstructions to projective surfaces admitting skew-symmetric Ricci tensor. *J. Geom. Phys.*, 76, pp. 192–199. doi:10.1016/j.geomphys.2013.10.019

[20] Ryden, B., 2017. *Introduction to Cosmology*, Cambridge University Press. doi:10.1017/9781316651087

[21] Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E., 2003. *Exact Solutions of Einstein's Field Equations*. Cambridge University Press. doi:10.1017/CBO9780511535185

[22] Weinberg, S., 1972. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons, New York.

[23] Zaeim, A., Cheshmavar J. and Musavi, M., 2024. Generalized Ricci Solitons on Homogeneous Siklos Space-times. *AUT Journal of Mathematics and Computing*, 5(2), pp. 131–142. doi:10.22060/AJMC.2023.22164.1138

Amirhesam Zaeim
Department of Mathematics,
Payame Noor University,
PO BOX 19395-4697,
Tehran, Iran
Email: zaeim@pnu.ac.ir

Ghasem Forozani
Department of Physics,
Payame Noor University,
PO BOX 19395-4697,
Tehran, Iran
Email: forozani@pnu.ac.ir

Maryam Dashti Darvishzadeh
Department of Physics,
Payame Noor University,
PO BOX 19395-4697,
Tehran, Iran
Email: m.dashti@student.pnu.ac.ir

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