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APPLICATION OF BLOCK PULSE FUNCTIONS TO ANALYTIC SOLUTION OF STOCHASTIC DELAYED DIFFERENTIAL EQUATIONS

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Abstract. The following paper introduces an analytic scheme for solving stochastic delayed differential equations via block palse functions. We present some preliminaries of block pulse functions and their related operational matrix of integration. This orthogonal basis function reduces the computation complexity of the stochastic delayed differential equations to a linear lower triangular system of algebraic equations which can be solved by forward substitution like LU decomposition method. Then the order of convergence of the suggested method has been discussed. Error analysis of two visual samples are mentioned to show the effectiveness of the suggested procedure.

1. Introduction

Delays occur naturally in just about every interaction of the real world, both natural and man-made in chemistry, economics, biology, ship stabilization, automatic steering, population dynamics and etc. Latency of a process or dead systems survey has been

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extended continuously over the past sixty years. Differential equations (DEs) in which the derivative of the unknown function at an exact moment are expressed with respect to the value of the previous time functions are called delay differential equations (DDEs) Smith (2100). On the other hand, the entrance of delay parameter to a stochastic differential equations (SDEs) leads to stochastic delayed differential equations(SDDEs). Such systems property are their dependency to a gaussian white noise, which follow probability laws Kuang 1993. SDDEs generate believable processes with real parameters. The occurrence of complex processes is often formulated by this subset of equations with continuous functions due to the ability to solve in infinite spaces of latency as well. Many studies has been done for the solution of delayed systems. A method for multi-delay systems based on hybrid block pulse function (BPFs) and Taylor series was presented by Marzban and Razaghi (2006). An approximated scheme for time-varying systems based on triangular functions (TFs) was introduced by Hosseini and Marzban (2011). The solution of DDEs via hybrid BPFs and Bernstein polynomials obtained by Behroozifar and Yousefi (2013).. A novel approach for time delayed fractional-order systems using BPFs has been proposed Tang et al.(2017). The solution of nonlinear SDDEs by $\theta-$ Maruyama methods was discussed Wang et al.(2015). In the following, we intend to solve SDDEs via an orthogonal function like Block pulse. Now think over the linear stochastic delay Volterra integral equations (SDVIEs),

$$\begin{cases} z(\vartheta) = z_0(\vartheta) + \int_0^{\vartheta} k_1(\varsigma,\vartheta)\; z(\varsigma-\tau)\, d\varsigma + \int_0^{\vartheta} k_2(\varsigma,\vartheta)\; z(\varsigma-\tau)\, dW(\varsigma), & \vartheta \in [0,T) \\ z(\vartheta) = \mu(\vartheta), & \vartheta \in [-\tau,0) \end{cases} \tag{1.1}$$

where $z(\vartheta)$, $z_0(\vartheta)$, $k_1(\varsigma,\vartheta)$,$k_2(\varsigma,\vartheta)$ and one - dimenional Wiener process named $W(\vartheta)$ for $\varsigma,\vartheta \in [0,T)$ are the stochastic processes defined on the same probability space with a filtration F_ϑ satisfying the usual conditions, i.e. the filtration $(F_\vartheta)_{\vartheta\geq 0}$ is right-continuous, F_ϑ, $(\vartheta \geq 0)$ and F_0 contains all P-null sets in F. $z(\vartheta)$ is unknown and the second integral is in $It\widehat{o}$ integral sense.$\mu(\vartheta)$ is a $F_{\vartheta_0}-$ measurable $C([-\tau;0],R)$- valued random variable such that $E\|\mu\|^2 < \infty, C([-\tau;0],R)$is the Banach space of all continuous paths from $[0,-\tau] \to R$ equipped with the supremum norm Buckwar(2000).

Our main goal is to transform the SDVIEs to algebraic representation by the usage of operational matrices of block pulse orthogonal function.
The remainder structure of this paper is as follows: Section 2 introduces an overview on block pulse delayed functions. Section 3 describes the main concept of our problem. Section 4 investigates error analysis and rate of convergence. Section 5 is devoted to numerical examples which depict the accuracy of the proposed approach. Section 6 finalizes the result of paper.

2. **An overview on delayed BPFs.**

Block pulse functions, have some efficient properties like disjointedness, orthogonality and completeness which various problems have been solved by them lonely or in hybrid manner.

2.1. **Time-varying BPFs operational matrix.** If we shift the m-set of $\Phi(\vartheta)$ by $\tau = (k+\lambda)h$, $k = 1, \ldots, m$, $h = \frac{T}{m}$, $0 \leq \lambda < 1$, let

$$\Phi(\vartheta - \tau) = [\Phi_1(\vartheta - \tau), \Phi_2(\vartheta - \tau), \ldots, \Phi_i(\vartheta - \tau)]^T.$$

where $[...]^T$, denotes transposition for $i = 1, ..., m$. We can derive the delay matrix as follows:

$$\begin{aligned}\Phi_i(\vartheta - \tau) = \Phi_i(\vartheta - kh - \lambda h) &= \begin{cases} 1 & (i-1)h \leq \vartheta - kh - \lambda h < ih \\ 0 & ow \end{cases}, \\ &= \begin{cases} 1 & (i+k+\lambda-1)h \leq \vartheta < (i+k+\lambda)\,h \\ 0 & ow \end{cases}, \\ &= \Phi_{i+k+\lambda}(\vartheta).\end{aligned}$$

Shift of a function is called delay and can be declared as

$$\Phi(\vartheta - \tau) = D.\,\Phi(\vartheta), \qquad 0 \leq \vartheta < T, \;\; \vartheta > \tau \tag{2.1}$$

Let D be a $m \times m$ operational time delayed matrix, which $1 - \lambda$ in the first row is $(k+1)\vartheta h-$ column component. Where

$$D=\begin{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}_{k\times 1} & \begin{bmatrix} 1-\lambda & \lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & \lambda & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \lambda \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}_{k\times k} \\ \begin{bmatrix} 0 \end{bmatrix}_{1\times 1} & \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{1\times k} \end{bmatrix}_{m\times m}.$$

In present work we set $\lambda = 0$ which was applied in Nouri and Maleknejad (2016).

2.2. **Functions approximation.** We can expand a $f(\vartheta)$ square integrable function with Lebesgue measure into a m-term BPFs series in $\vartheta \in [0, T)$ as

$$f(\vartheta) \simeq [f_0, f_1, ..., f_m] .\Phi(\vartheta),$$

where

$$f_i = \frac{1}{h} \int_{(i-1)h}^{ih} f(\vartheta)\, d\vartheta, \quad i = 1, ..., m.$$

Let $k(\varsigma, \vartheta) \in L^2([0, T_1), [0, T_2))$ and be expanded in terms of BPFs such as

$$k_q(\varsigma, \vartheta) \approx \Psi^T(\varsigma).\ K_q.\Phi(\vartheta) = \Phi^T(\vartheta).K_q^T\ .\Psi(\varsigma), \qquad q = 1, 2.$$

where $\Phi(\vartheta) = [...]_{m_1\times 1}$,$\Psi(\varsigma) = [...]_{m_2\times 1}$ are BPFs vectors respectively. And

$$K_q = (k_{ij}),\ i = 1, 2, ..., m_1; j = 1, 2, ..., m_2$$

are $m_1 \times m_2$ BPF coefficients matrix with

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1}\int_0^{T_2} k(\varsigma, \vartheta).\Psi_i(\varsigma).\Phi_j(\vartheta)\ d\vartheta\, d\varsigma,$$

For simplicity, we set $h_1 = h_2 = h$, $T_1 = T_2 = T$, $m_1 = m_2 = m$ and

$$k_{ij} = \frac{1}{h^2} \int_{(i-1)h}^{ih} \int_{(j-1)h}^{jh} k(\varsigma, \vartheta)\ d\vartheta\, d\varsigma, \quad i = 1, ..., m.$$

2.3. **Operational matrix of BPFs respect to deterministic and stochastic form.** Computation of this section is completely explained in Maleknejad et al.(2012).

i.Deterministic operational matrix of BPFs,

$$\int_{\tau}^{\vartheta} \Phi(\varsigma)\ d\varsigma = P.\,\Phi\left(\vartheta\right), \tag{2.2}$$

where

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m}.$$

ii. Stochastic operational matrix of BPFs,

$$\int_{\tau}^{\vartheta} \Phi(\varsigma)\ dW(\varsigma) = P_{\varsigma}.\Phi\left(\vartheta\right), \tag{2.3}$$

which stochastic operational matrix of integration P_{ς} is defined through,

$$P_{\varsigma} = \begin{bmatrix} W(\frac{h}{2}) & W(h) & W(h) & \cdots & W(h) \\ 0 & W(\frac{3h}{2}) - W(h) & W(2h) - W(h) & \cdots & W(2h) - W(h) \\ 0 & 0 & W(\frac{5h}{2}) - W(2h) & \cdots & W(3h) - W(h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & W(\frac{(2m-1)h}{2}) - W((m-1)h) \end{bmatrix}_{m\times m},$$

Note that: $\left(\int_0^{\tau} \Psi(\varsigma).\ \Psi^T(\varsigma)\ d\varsigma\right) = \left(\int_0^{\tau} \Psi(\varsigma).\ \Psi^T(\varsigma)\ dW(\varsigma)\right) = 0.$

iii.

$$\begin{aligned} &\left(\int_{\tau}^{\vartheta} \Phi(\varsigma).\ \Phi^T(\varsigma)\ d\varsigma\right).K_1\ .\Phi\left(\vartheta\right) = B_1.\,\Phi\left(\vartheta\right), \\ &\left(\int_{\tau}^{\vartheta} \Phi(\varsigma).\ \Phi^T(\varsigma)\ dW(\varsigma)\right).K_2.\Phi(\vartheta) = B_2.\Phi(\vartheta), \end{aligned} \tag{2.4}$$

where

$$B_1 = \frac{h}{2}\begin{bmatrix} k_{11} & 2k_{12} & 2k_{13} & \cdots & 2k_{1m} \\ 0 & k_{22} & 2\,k_{23} & \cdots & 2k_{2m} \\ 0 & 0 & k_{33} & \cdots & 2k_{3m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{mm} \end{bmatrix},$$

and

$$B_2 = \begin{cases} (k_{ij})^2.\,(W(ih) - W((i-1)h)\,) & i < j, \\ (k_{ii})^2.\left(W(\frac{(2i-1)h}{2}) - W((i-1)h)\right) & i = j, \\ 0 & i > j. \end{cases} \qquad [...]_{m\times m} \quad i,j = 1,2,...,m.$$

3. **Problem statement**

Consider a simplest time delayed ODE,

$$\begin{cases} \frac{dz}{d\vartheta} = \alpha(\varsigma,\vartheta).\,z(\vartheta-\tau), & \vartheta \in [0,T)\,, \\ z(\vartheta) = \mu(\vartheta) & \vartheta \in [-\tau,0)\,, \end{cases} \tag{3.1}$$

where $\alpha\,(\vartheta,\varsigma) > 0$, defines growth rate and a system with positive feedback, $z(\vartheta-\tau)$ indicates the trajectory of the solution in the past. Step-wise manner called method of steps are mostly used to analytically solve DDEs. $\mu\,(\vartheta)$ is the banach space of all continuous path from $[-\tau,0) \to R$ equipped with the supremum norm. If above equation is exposed to a white gaussian noise named $\xi(\vartheta)$ by $b(\vartheta,\varsigma)$ noise intensity, it may be affected in two case Khodabin and Kiaee (2011):

i:

$$\frac{dz}{d\vartheta} = \alpha(\varsigma,\vartheta).\,(z(\vartheta-\tau) + b(\varsigma,\vartheta).\,\xi\,(\vartheta))$$

and

$$dz = \alpha(\varsigma,\vartheta).z(\vartheta-\tau)d\vartheta + \alpha(\varsigma,\vartheta).b(\varsigma,\vartheta)\,dW(\vartheta)\,,$$

that with the same trend leads to additive noise.

$$z(\vartheta) = z_0(\vartheta) + \int_0^{\vartheta} k_1(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, d\varsigma + \int_0^{\vartheta} k_2(\varsigma, \vartheta)\; dW(\varsigma).$$

ii:

$$\tfrac{dz}{d\vartheta} = (\alpha(\varsigma, \vartheta) + b(\varsigma, \vartheta).\, \xi(\vartheta))\; .z(\vartheta - \tau),$$

which $\xi(\vartheta) = \frac{dW(\vartheta)}{d\vartheta}$,

$$dz = \alpha(\varsigma, \vartheta) \,.\, z(\vartheta - \tau)\, d\vartheta + b(\varsigma, \vartheta).\, z(\vartheta - \tau)\, dW(\vartheta),$$

with integration in interval $\vartheta \in [a, b)$, we arrive,

$$z(\vartheta) = z_0(\vartheta) + \int_a^{b} \alpha(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, d\varsigma + \int_a^{b} b(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, dW(\varsigma)$$

In other words,

$$z(\vartheta) = z_0(\vartheta) + \int_a^{b} k_1(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, d\varsigma + \int_a^{b} k_2(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, dW(\varsigma).$$

that it leads to multiplicative noise. □

Generally we assume $a = 0$ and $b = 1$ to simplify the usage of block pulse functions in $\vartheta \in [0, T)$. In case $a \neq 0$. Set $\varsigma = \frac{\vartheta - a}{b - a} T$ where $T = m.h$.

3.1. **Solving SDDEs via block pulse operational matrices with additive noise.** Below, we deal with a linear stochastic Volterra integral equation with single time delay, we intend to obtain block pulse coefficients of the $z(\vartheta)$ in the interval $\vartheta \in [0, T]$,

(3.2)

$$\begin{cases} z(\vartheta) = z_0(\vartheta) + \int_0^{\vartheta} k_1(\varsigma, \vartheta) \,.\, z(\varsigma - \tau)\, d\varsigma + \int_0^{\vartheta} k_2(\varsigma, \vartheta)\, dW(\varsigma) & \tau \in [0, T), \\ z(0) = z_0, & \\ z(\vartheta) = \mu(\vartheta) & -\tau \leq \vartheta < 0. \end{cases}$$

where z_0 is a constant specified vector, and $k_1(\varsigma,\vartheta)$, $k_2(\varsigma,\vartheta)$ are known matrices and $\mu(\vartheta)$, is an arbitrary and initial history known function. $z(\vartheta), z_0(\vartheta), k_1(\varsigma,\vartheta)$ and $k_2(\varsigma,\vartheta)$ functions are approximated by BPFs as mentioned below from (3.3)-(3.7)

$$z(\vartheta) \simeq \bar{z}(\vartheta) \approx Z^T.\Phi(\vartheta) = \Phi^T(\vartheta)\, Z_\vartheta, \qquad \tau \le \vartheta < T, \tag{3.3}$$

$$z_0(\vartheta) \simeq Z_0^T.\Phi(\vartheta) = \Phi^T(\vartheta)\, Z_0, \tag{3.4}$$

$$k_1(\varsigma,\vartheta) \simeq \Psi^T(\varsigma)\,.K_1.\,\Phi(\vartheta) = \Phi^T(\vartheta)\,.K_1^T\,.\,\Psi(\varsigma), \tag{3.5}$$

$$k_2(\varsigma,\vartheta) \simeq \Psi^T(\varsigma)\,.K_2\,.\Phi(\vartheta) = \Phi^T(\vartheta)\,.K_2^T\,.\,\Psi(\varsigma), \tag{3.6}$$

$$z(\vartheta-\tau) = \begin{cases} \mu(\vartheta-\tau), & -\tau \le \ \vartheta-\tau\ < 0, \\ Z^T.\Phi(\vartheta-\tau) = Z^T.D.\Phi(\vartheta), & 0\ \le\ \vartheta-\tau\ < T-\tau,, \end{cases} \tag{3.7}$$

Where the vectors Z, Z_0 and matrices K_1, K_2 are BPF coefficients of $z(\vartheta)$, $z_0(\vartheta)$, $k_1(\varsigma,\vartheta)$ and $k_2(\varsigma,\vartheta)$, respectively. With replacement of above approximation (3.3)-(3.7) into (3.2), we arrive to

$$\begin{cases} Z^T.\Phi(\vartheta) \simeq Z_0^T.\Phi(\vartheta) + \int_0^\tau \mu(\varsigma-\tau).\Psi^T(\varsigma)\,.K_1.\,\Phi(\vartheta)\,d\varsigma + \int_\tau^\vartheta Z^T.\Psi(\varsigma-\tau).\Psi^T(\varsigma)\,.K_1.\,\Phi(\vartheta)\,d\varsigma \\ \qquad\qquad + \int_0^\vartheta \Psi^T(\varsigma)\,.K_2.\,\Phi(\vartheta)\,dW(\varsigma) \end{cases},$$

then

$$\begin{cases} Z^T.\Phi(\vartheta) \simeq Z_0^T.\Phi(\vartheta) + \left(\int_0^\tau \mu(\varsigma-\tau).\Psi^T(\varsigma)\ \ d\varsigma\right).K_1.\Phi(\vartheta) + \left(\int_\tau^\vartheta Z^T.D.\,\Psi(\varsigma).\Psi^T(\varsigma)\ \ d\varsigma\right).K_1.\Phi(\vartheta) \\ \qquad\qquad + \left(\int_0^\vartheta \Psi^T(\varsigma)\ \ dW(\varsigma)\right).K_2.\Phi(\vartheta) \end{cases},$$

and

$$Z^T.\Phi(\vartheta) \simeq Z_0^T.\Phi(\vartheta) + I_1.K_1.\Phi(\vartheta) + Z^T.D.\left(\int_\tau^\vartheta \Psi(\varsigma).\Psi^T(\varsigma)\ \ d\varsigma\right)K_1.\,\Phi(\vartheta) + I_2.K_2.\Phi(\vartheta),$$

consider that

$$I_1 = \int_0^{\tau} \mu\,(\varsigma - \tau)\,.\Psi^T\,(\varsigma)\;\; d\varsigma,$$
$$I_2 = \int_0^{\vartheta} \Psi^T\,(\varsigma)\;\; dW(\varsigma),$$

and

$$\left(\int_{\tau}^{\vartheta} \Psi(\varsigma).\Psi^T\,(\varsigma)\;\; d\varsigma\right).K_1.\,\Phi(\vartheta) = B_1.\Phi(\vartheta),$$

and B_1 was defined in (2.4) as well. So

$$Z^T.\Phi(\vartheta) \simeq Z_0^T.\Phi(\vartheta) + I_1.K_1.\,\Phi(\vartheta) + Z^T.D.B_1.\,\Phi(\vartheta) + I_2.K_2.\,\Phi(\vartheta),$$

If we multiply $\Phi^T(\vartheta)$ on both sides, we arrive to

$$Z^T.\,(I - D.B_1\,) \simeq Z_0^T + I_1.K_1 \; + I_2.K_2, \tag{3.8}$$

So by setting $U^T = I - D.B_1, U_0^T = Z_0^T + I_1.K_1 + I_2.K_2$ and replacing $\simeq$ by $=$, we have

$$U.Z = U_0. \tag{3.9}$$

Unknown function $z(\vartheta)$ is obtained simply by substituting the approximated BPF coefficients $Z(\vartheta)$ in (3.3).

3.2. **Solving SDDEs via block pulse operational matrices with multiplicative noise.** Subsequently, we obtain block pulse coefficient of $z\,(\vartheta)$ in interval $\vartheta \in [a, b]$ the same as previous section.

(3.10)
$$\begin{cases} z(\vartheta) = z_0(\vartheta) + \int_0^{\vartheta} k_1(\varsigma, \vartheta)\,.\,z(\varsigma - \tau)\,d\varsigma + \int_0^{\vartheta} k_2(\varsigma, \vartheta)\,.\,z(\varsigma - \tau)\,dW(\varsigma) & \tau \in [0, T), \\ z(0) = z_0, & \\ & \\ z(\vartheta) = \mu\,(\vartheta) & -\tau \leq \vartheta < 0. \end{cases}$$

With substituting function's approximation into (3.3)-(3.7) in (3.10), we arrive at,

$$\begin{cases} Z^T.\Phi(\vartheta) = Z_0^T.\Phi(\vartheta) + \int\limits_0^\tau \mu\left(\varsigma-\tau\right).\Psi^T\left(\varsigma\right)\ .K_1.\,\Phi(\vartheta)\,d\varsigma + \int\limits_\tau^\vartheta Z^T.\Psi(\varsigma-\tau).\Psi^T\left(\varsigma\right)\ .K_1.\,\Phi(\vartheta)\,d\varsigma \\ \qquad\qquad + \int\limits_0^\tau \mu\left(\varsigma-\tau\right).\Psi^T\left(\varsigma\right)\ .K_2.\,\Phi(\vartheta)\,dW(\varsigma) + \int\limits_\tau^\vartheta Z^T.\Psi(\varsigma-\tau).\Psi^T\left(\varsigma\right)\ .K_2.\,\Phi(\vartheta)\,dW(\varsigma) \end{cases},$$

and

$$\begin{cases} Z^T.\Phi(\vartheta) = Z_0^T.\Phi(\vartheta) + \left(\int\limits_0^\tau \mu\left(\varsigma-\tau\right).\Psi^T\left(\varsigma\right)\ \ d\varsigma\right).K_1.\,\Phi(\vartheta) + Z^T.D\left(\int\limits_\tau^\vartheta \Psi(\varsigma).\Psi^T\left(\varsigma\right)\ \ d\varsigma\right).K_1.\,\Phi(\vartheta) \\ \qquad\qquad \left(\int\limits_0^\tau \mu\left(\varsigma-\tau\right).\Psi^T\left(\varsigma\right)\ \ dW(\varsigma)\right).K_2.\,\Phi(\vartheta) + Z^T.D.\left(\int\limits_\tau^\vartheta \Psi(\varsigma).\Psi^T\left(\varsigma\right)\ dW(\varsigma)\right).K_2.\,\Phi(\vartheta) \end{cases},$$

let that,

$$I_3 = \int\limits_0^\tau \mu\left(\varsigma-\tau\right).\Psi^T\left(\varsigma\right)\ \ dW(\varsigma),$$

so,

$$Z^T.\Phi(\vartheta) \simeq Z_0^T.\Phi(\vartheta) + I_1.K_1.\,\Phi(\vartheta) + I_3.K_2.\,\Phi(\vartheta) + Z^T.D.B_1.\,\Phi(\vartheta) + Z^T.D.\,B_2.\,\Phi(\vartheta),$$

by multiplying $\Phi^T(\vartheta)$ and rearranging the equation, we have

$$Z^T\left(I - D.\left(B_1 + \ B_2\right)\right) \simeq Z_0^T + I_1.K_1\ + I_2.K_2,$$

hence by setting $R^T = I - D.\left(B_1 + \ B_2\right)$, $R_0^T = Z_0^T + I_1.K_1\ + I_2.K_2$ and replacing $\simeq$ by $=$,we have

$$R.Z = R_0. \tag{3.11}$$

which (3.9) and (3.11) are linear systems of lower triangular coefficients matrix. Such systems present the approximate block pulse coefficient of the unknown stochastic processes $z(\vartheta)$. This complete the proofs. □

4. **Error analysis and rate of convergence**

This part is allocated to the convergence rate proof. The results show a good level of accuracy of order $O(h)$. We utilize the two first theorems of Nouri and Maleknejad (2016). below and assert Theorems (4.3) and (4.4).

Theorem 4.1. *Assume the square integrable function $z_0(\vartheta)$ in the interval $[0,1)$,which is arbitrary and real bounded and $e(\vartheta) = z_0(\vartheta) - (\bar{z}_0)_m(\vartheta)$, $\vartheta \in I = [0,1)$, such that;*

$$(\bar{z}_0)_m(\vartheta) = \sum_{i=1}^{m} (z_0)_i . \varphi_i^{(m)}(\vartheta)$$

is the block pulse series of $z_0(\vartheta)$ .Therefore,

$$\|e(\vartheta)\| \leq \frac{h}{2\sqrt{3}} \sup_{\vartheta \in I} |({z_0}')_i(\vartheta)| .$$

Theorem 4.2. *Suppose that $f(\varsigma, \vartheta) \in L^2([0,1) \times [0,1))$ and*

$$e(\varsigma, \vartheta) = f(\varsigma, \vartheta) - \bar{f}_m(\varsigma, \vartheta), \qquad (\varsigma, \vartheta) \in D = [0,1) \times [0,1),$$

which $\bar{f}_m(\varsigma, \vartheta) = \sum_{i=1}^{m} \sum_{j=1}^{m} f_{ij}\, \psi_i^{(m)}(\varsigma)\, .\varphi_j^{(m)}(\vartheta)$ is the block pulse series of $f(\varsigma, \vartheta)$. Then,

$$\|e(\varsigma, \vartheta)\| \leq \frac{h}{2\sqrt{3}} \left(\sup_{(\varsigma,\vartheta)\in D} |{f'}_{\varsigma}(\varsigma, \vartheta)|^2 + \sup_{(\varsigma,\vartheta)\in D} |{f'}_{\vartheta}(\varsigma, \vartheta)|^2 \right)^{\frac{1}{2}} .$$

Or

$$\|e(\varsigma, \vartheta)\| = O(h).$$

Theorem 4.3. *Suppose that $z(\vartheta - \tau)$ is an arbitrary real bounded and delayed function, which is square integrable in the interval $[0,1)$.*

$$e(\vartheta - \tau) = z(\vartheta - \tau) - \overline{z}(\vartheta - \tau)$$

Which $\bar{z}_m(\vartheta-\tau)=\sum_{i=1}^{m}(z)_i\Phi_i^{(m)}(\vartheta-\tau)$ *is the block pulse series of* $z(\vartheta-\tau)$ *. Then,*

$$\|e(\vartheta-\tau)\|=\frac{h}{2\sqrt{3}}\sup_{\vartheta\in I}|z'(\eta_i)|,\quad or\quad \|e(\vartheta-\tau)\|=O(h),\ \vartheta\in I$$

Proof.

$$e_i(\vartheta-\tau)=\begin{cases}z(\vartheta-\tau)-\overline{z}_i(\vartheta-\tau), & \vartheta\in D_i,\\ 0, & \vartheta\in I\setminus D_i,\end{cases}$$

Where $D_i=\left\{\vartheta-\tau:(i-1)h\le\vartheta-\tau\le ih, h=\frac{1}{m}\ ;\ i=1,2,...,m\right\}$*. Then,*

$$e_i(\vartheta-\tau)=z(\vartheta-\tau)-\sum_{i=1}^{m}z_i\Phi_i^{(m)}(\vartheta-\tau)=z(\vartheta-\tau)-\frac{1}{h}\int_{(i-1)h}^{ih}z(\varsigma)d\varsigma=\frac{1}{h}\left(\int_{(i-1)h}^{ih}z(\vartheta-\tau)-z(\vartheta-\tau)d\varsigma\right)$$

Now by mean value theorem, we get,

$$=\frac{z'(\eta_i)}{h}\left(\int_{(i-1)h}^{ih}(\vartheta-\varsigma)\,d\tau\right)=\frac{z'(\eta_i)}{h}\left(\vartheta\varsigma-\frac{\varsigma^2}{2}\right)\Big|_{(i-1)h}^{ih}=z'(\eta_i)\left(\vartheta+\left(\frac{1}{2}-i\right)h\right)$$

$$\vartheta,\ \eta_i\in D_i,\qquad i=1,2,\ldots,m.$$

Then,

$$\|e_i(\vartheta-\tau)\|^2=\int_{(i-1)h}^{ih}|e_i(\vartheta-\tau)|^2d\vartheta=\int_{(i-1)h}^{ih}\left(z'(\eta_i)\left(\vartheta+\left(\frac{1}{2}-i\right)h\right)\right)^2d\vartheta=\frac{h^3}{12}(z'(\eta_i))^2$$

Such that $\eta_i\in D_i$ *,* $i=1,2,...,m$*. Consequently*

$$\|e(\vartheta-\tau)\|^2=\int_0^1|e(\vartheta-\tau)|^2d\vartheta=\int_0^1\left(\sum_{i=1}^{m}e_i(\vartheta-\tau)\right)^2d\vartheta=$$

$$\int_0^1\left(\sum_{i=1}^{m}e_i^2(\vartheta-\tau)+2\sum_{i<j}^{m}e_i(\vartheta-\tau)e_j(\vartheta-\tau)\right)\,d\vartheta=\sum_{i=1}^{m}\left(\int_0^1e_i^2(\vartheta-\tau)d\vartheta\right)=$$

$$\frac{h^3}{12}\sum_{i=1}^{m}\left(z'(\eta_i)\right)^2\le\frac{h^2}{12}\sup|z'(\eta_i)|^2.$$

Or

$$\|e(\vartheta-\tau)\| = O(h).$$

Theorem 4.4. *Let $z(\vartheta)$ and $\bar{z}(\vartheta)$ be solution of equation (3.2) and (3.10), respectively, and let $\|z(\vartheta)\| < C$ and $\|k_i\| < C$ for $i = 1, 2$. Then*

$$(\|z(\vartheta) - \bar{z}(\vartheta)\|) \leq O(h) \qquad \vartheta \in [0, 1).$$

Proof. *Let*

$$\begin{cases} z(\vartheta) - \bar{z}(\vartheta) = z_0(\vartheta) - \bar{z}_0(\vartheta) + \\ \\ \int_0^\vartheta k_1(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_1(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\, d\varsigma + \int_0^\vartheta k_2(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_2(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\, dW(\varsigma) \end{cases}$$

With using $(a+b)^2 \leq 3(a^2+b^2+c^2)$ we have,

$$\begin{cases} E|z(\vartheta) - \bar{z}(\vartheta)|^2 \leq 3\left(E|z_0(\vartheta) - \bar{z}_0(\vartheta)|^2\right. \\ \\ \left. \left|E\int_0^\vartheta k_1(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_1(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\, d\varsigma\right|^2 + E\left|\int_0^\vartheta k_2(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_2(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\, dW(\varsigma)\right|^2\right) \end{cases}$$

Based on Itô isometry property and Fubini theorem[13], *we get*

$$E\left|\int_0^\vartheta k_2(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_2(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\, dW(\varsigma)\right|^2 = \int_0^\vartheta E\left|\, k_2(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_2(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\right|^2 d\varsigma$$

Then, based on $\|X\|^2 = \mathbb{E}\,|X|^2$ as defined in Kanwal (2013).

$$\begin{cases} \|z(\vartheta) - \bar{z}(\vartheta)\|^2 \leq 3\Big(\|z_0(\vartheta) - \bar{z}_0(\vartheta)\|^2 + \left\|\int_0^\vartheta \left(k_1(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_1(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\right) d\varsigma\right\|^2 \\ \qquad\qquad + \int_0^\vartheta \left\|k_2(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_2(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\right\|^2 d\varsigma\Big) \end{cases}$$

since,

$$\begin{cases} \left\|k_i(\varsigma,\vartheta)\, z(\vartheta-\tau) - \bar{k}_i(\varsigma,\vartheta)\, \bar{z}(\vartheta-\tau)\right\|^2 = \\ 2\|k_i(\varsigma,\vartheta)\,(z(\vartheta-\tau) - \bar{z}(\vartheta-\tau))\|^2 + 2\left\|\left(k_i(\varsigma,\vartheta) - \bar{k}_i(\varsigma,\vartheta)\right)\bar{z}(\vartheta-\tau)\right\|^2 \\ \leq C.\|z(\vartheta-\tau) - \bar{z}(\vartheta-\tau)\|^2 + C.\left\|k_i(\varsigma,\vartheta) - \bar{k}_i(\varsigma,\vartheta)\right\|^2 \;\; i = 1, 2. \end{cases}$$

and,

$$\begin{cases} \left\|k_i(\varsigma,\vartheta)-\bar{k}_i(\varsigma,\vartheta)\right\|^2 = O(h^2) \quad i=1,2. \\ \|z(\vartheta-\tau)-\bar{z}(\vartheta-\tau)\|^2 = O(h^2) \end{cases}$$

we have,

$$\begin{cases} \|z(\vartheta)-\bar{z}(\vartheta)\|^2 \leq 3\left(\|z_0(\vartheta)-\bar{z}_0(\vartheta)\|^2 + \right. \\ \left. C.\|z(\vartheta-\tau)-\bar{z}(\vartheta-\tau)\|^2 + C.\left\|k_1(\varsigma,\vartheta)-\bar{k}_1(\varsigma,\vartheta)\right\|^2 + C.\left\|k_2(\varsigma,\vartheta)-\bar{k}_2(\varsigma,\vartheta)\right\|^2\right) \end{cases}$$

so,

$$\|z(\vartheta)-\bar{z}(\vartheta)\|^2 \leq O(h^2), \qquad \|z(\vartheta)-\bar{z}(\vartheta)\| \leq O(h).$$

This completes the proof. □

5. **Illustrative examples**

The following examples are devoted to demonstrate the theoretical results as analytic or approximated solutions stated in section 3 and 4. The related computations were performed using *Matlab* software.

Consider, below assumptions:

z_i: Block pulse coefficient of analytic solution.

$\bar{z}_i$: Block pulse coefficient of approximated method.

e: Absolute error computed by $\|e\|_\infty = \max_{1\leq i\leq m}|z_i-\bar{z}_i|$.

n: Number of iterations.

$\bar{z}_e$: Error of means.

s_e: Standard deviation of error.

UB: Upper bound.

LB: Lower bound.

Time delayed equations are mostly solved analytically in a step-wise procedure called the method of steps with given initial conditions.

Example 5.1. Assume a linear stochastic Volterra integral equation with an additive

noise input and a constant time delay given by

$$\begin{cases} z(\vartheta) = z(0) + \int_0^{\vartheta} 4\, z\big(\varsigma - \frac{1}{4}\big)\, d\varsigma + 4b \int_0^{\vartheta} dW(\varsigma), & \vartheta \in [0,1), \\ z(\vartheta) = 1, & \vartheta \in \big[-\frac{1}{4}, 0\big), \\ z(0) = 1. \end{cases} \tag{5.1}$$

Here, $z(\vartheta)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $W(\vartheta)$ denotes a Wiener process. The analytical solution is obtained using the method of steps. Consider an ordinary differential equation with single time delay,

$$\frac{dz(\vartheta)}{d\vartheta} = 4\, z\big(\vartheta - \tfrac{1}{4}\big)\,. \tag{5.2}$$

If the time-delayed variable is affected by white noise, then the stochastic delay differential equation is given by

$$\frac{dz(\vartheta)}{d\vartheta} = 4\left(z\left(\vartheta - \frac{1}{4}\right) + b\,\zeta(\vartheta)\right). \tag{5.3}$$

Its analytical solution can be obtained by the stepwise method.

Step (i): $0 \le \vartheta < \frac{1}{4}$, then $-\frac{1}{4} \le \vartheta - \frac{1}{4} < 0$ and $z(\vartheta - \frac{1}{4}) = 0$. Hence,

$$z(\vartheta) = 1 + 4b\, W(\vartheta), \quad \vartheta \in [0, \tfrac{1}{4}), \qquad z\big(\tfrac{1}{4}\big) = 1 + 4b\, W\big(\tfrac{1}{4}\big)\,.$$

Step (ii): $\frac{1}{4} \le \vartheta < \frac{2}{4}$, so $0 \le \vartheta - \frac{1}{4} < \frac{1}{4}$ and $z(\vartheta - \frac{1}{4}) = 1 + 4b\, W(\vartheta - \frac{1}{4})$. Then

$$z(\vartheta) = 1 + 4(\vartheta - \tfrac{1}{4}) + 4b\left[4\int_{\frac{1}{4}}^{\vartheta} W(\varsigma - \tfrac{1}{4})\, d\varsigma + W(\vartheta)\right], \quad \vartheta \in [\tfrac{1}{4}, \tfrac{1}{2}),$$

and

$$z\big(\tfrac{2}{4}\big) = 2 + 16b \int_{\frac{1}{4}}^{\frac{2}{4}} W(\varsigma - \tfrac{1}{4})\, d\varsigma + 4b\, W\big(\tfrac{1}{2}\big)\,.$$

Step (iii): $\frac{2}{4} \le \vartheta < \frac{3}{4}$, $\frac{1}{4} \le \vartheta - \frac{1}{4} < \frac{2}{4}$, $z(\vartheta - \frac{1}{4}) = 1 + 4(\vartheta - \frac{1}{2}) + 16b \int_{\frac{1}{4}}^{\vartheta - \frac{1}{4}} W(\varsigma - \frac{1}{4})\, d\varsigma + 4bW(\vartheta - \frac{1}{4})$. Then

$$z(\vartheta) = 1 + 4(\vartheta - \tfrac{1}{4}) + 8(\vartheta - \tfrac{1}{2})^2 + 4b\Big[16(\vartheta - \tfrac{1}{2}) + \int_{\frac{1}{4}}^{\vartheta - \frac{1}{4}} W(\varsigma - \frac{1}{4})\, d\varsigma + 4\int_{\frac{1}{2}}^{\vartheta} W(u - \frac{1}{4})\, du$$
$$+ 4\int_{\frac{1}{4}}^{\frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma + W(\vartheta)\Big], \quad \vartheta \in [0.5, 0.75),$$

$$z\big(\tfrac{3}{4}\big) = \frac{7}{2} + 16b + 4b\int_{\frac{1}{4}}^{\frac{2}{4}} W(\varsigma - \frac{1}{4})\, d\varsigma + 16b\int_{\frac{1}{2}}^{\frac{3}{4}} W(u - \frac{1}{4})\, du + 16b\int_{\frac{1}{4}}^{\frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma + 4bW\left(\frac{3}{4}\right).$$

Step (iv): $\frac{3}{4} \le \vartheta < 1$, $\frac{2}{4} \le \vartheta - \frac{1}{4} < \frac{3}{4}$:

$$z(\vartheta) = 1 + 4(\vartheta - \tfrac{1}{2}) + 8(\vartheta - \tfrac{3}{4})^2 + \frac{32}{3}(\vartheta - \tfrac{3}{4})^3 + 4b\Big[8(\vartheta - \tfrac{3}{4}) \int_{\frac{1}{4}}^{\vartheta - \frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma + 16(\vartheta - \tfrac{3}{4}) \int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(u - \frac{1}{4})\, du + 4 \int_{\frac{3}{4}}^{\vartheta} W(v - \frac{1}{4})\, dv + 8(1 + 2(\vartheta - \tfrac{3}{4})) \int_{\frac{1}{4}}^{\frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma + W(\vartheta)\Big], \quad \vartheta \in [0.75, 1).$$

As a result, the solution of the stochastic delay differential equation with additive coefficient is obtained as above.

$$z(\vartheta) = \begin{cases} 1 + 4b\, W(\vartheta), & 0 \le \vartheta < \frac{1}{4}, \\ 1 + 4(\vartheta - \frac{1}{4}) + 4b\Big(4 \int_{\frac{1}{4}}^{\vartheta} W(\varsigma - \frac{1}{4})\, d\varsigma + W(\vartheta)\Big), & \frac{1}{4} \le \vartheta < \frac{2}{4}, \\ 1 + 4(\vartheta - \frac{1}{4}) + 8(\vartheta - \frac{1}{2})^2 + 4b\Big[16(\vartheta - \frac{1}{2}) + \int_{\frac{1}{4}}^{\vartheta - \frac{1}{4}} W(\varsigma - \frac{1}{4})\, d\varsigma \\ \quad +4 \int_{\frac{1}{2}}^{\vartheta} W(u - \frac{1}{4})\, du + 4 \int_{\frac{1}{4}}^{\frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma + W(\vartheta)\Big], & \frac{2}{4} \le \vartheta < \frac{3}{4}, \\ 1 + 4(\vartheta - \frac{1}{4}) + 8(\vartheta - \frac{1}{2})^2 + \frac{32}{3}(\vartheta - \frac{3}{4})^3 + 4b\Bigg[\frac{4^3}{2}(\vartheta - \frac{3}{4}) \int_{\frac{1}{4}}^{\vartheta - \frac{1}{2}} W(\varsigma - \frac{1}{4})\, d\varsigma \\ \quad +4^2(\vartheta - \frac{3}{4}) \int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(u - \frac{1}{4})\, du + 4 \int_{\frac{3}{4}}^{\vartheta} W(v - \frac{1}{4})\, dv \\ \quad +8(1 + 2(\vartheta - \frac{3}{4})) \int_{\frac{1}{4}}^{\frac{2}{4}} W(\varsigma - \frac{1}{4})\, d\varsigma + 4 \int_{\frac{1}{2}}^{\frac{3}{4}} W(u - \frac{1}{4})\, du + W(\vartheta)\Bigg], & \frac{3}{4} \le \vartheta < 1. \end{cases}$$

Example 5.2. Consider a linear stochastic Volterra integral equation with multiplicative noise input and time-varying delay as follows,

$$\begin{cases} z(\vartheta) = z(0) + \int_0^{\vartheta} 4\ z(\varsigma - \frac{1}{4})\, d\varsigma + 4b \int_0^{\vartheta}\ z(\varsigma - \frac{1}{4})\, dW(\varsigma), & \vartheta \in [0, 1) \\ z(\vartheta) = 1, & \vartheta \in \big[-\frac{1}{4}, 0\big) \\ z(0) = 1. \end{cases}$$

Where $z(\vartheta)$ is an unknown stochastic process in the probability space, (Ω, F, P), and $W(\vartheta)$ is a Wiener process. If the coefficient (growth rate) of the unknown delay function $z(\vartheta - \frac{1}{4})$ with the noise intensity b is affected by white Gaussian noise, the stochastic differential equation with multiplicative noise is obtained. In order to get the analytical solution of this equation by step method.

Stepwise solution: Consider

$$\begin{aligned} dz(\vartheta) &= 4\, z(\vartheta - \tfrac{1}{4})\, d\vartheta + 4b\, z(\vartheta - \tfrac{1}{4})\, dW(\vartheta), \quad \vartheta \in [0,1), \\ z(\vartheta) &= 0, \quad -\tfrac{1}{4} \le \vartheta < 0, \\ z(0) &= 1. \end{aligned}$$

Step (i): $0 \le \vartheta < \frac{1}{4}$, $z(\vartheta - \frac{1}{4}) = 0$:

$$z(\vartheta) - z(0) = 0 \Rightarrow z(\vartheta) = 1, \quad z(\tfrac{1}{4}) = 1.$$

Step (ii): $\frac{1}{4} \le \vartheta < \frac{2}{4}$, $z(\vartheta - \frac{1}{4}) = 1$:

$$z(\vartheta) = 1 + 4(\vartheta - \tfrac{1}{4}) + b\big(W(\vartheta) - W(\tfrac{1}{4})\big), \quad z(\tfrac{2}{4}) = 2 + b\big(W(\tfrac{2}{4}) - W(\tfrac{1}{4})\big).$$

Step (iii): $\frac{2}{4} \le \vartheta < \frac{3}{4}$, $z(\vartheta - \frac{1}{4}) = 1 + 4(\vartheta - \frac{1}{2}) + b(W(\vartheta - \frac{1}{4}) - W(\frac{1}{4}))$:

$$\begin{aligned} z(\vartheta) &= 1 + 4(\vartheta - \tfrac{1}{4}) + 8(\vartheta - \tfrac{1}{2})^2 \\ &\quad + 4b\left[\int_{\frac{1}{2}}^{\vartheta} W(\varsigma - \tfrac{1}{4})\, d\varsigma - \int_{\frac{1}{2}}^{\vartheta} W(\varsigma)\, d\varsigma + \vartheta\big(W(\vartheta) - W(\tfrac{1}{4})\big)\right] \\ &\quad + b\left[b\int_{\frac{1}{2}}^{\vartheta} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) - (1 + bW(\tfrac{1}{4}))W(\vartheta) + W(\tfrac{1}{4})(1 + bW(\tfrac{1}{2}))\right], \\ z(\tfrac{3}{4}) &= \tfrac{7}{2} + 4b\left[\int_{\frac{2}{4}}^{\frac{3}{4}} (W(\varsigma - \tfrac{1}{4}) - W(\varsigma))\, d\varsigma + \frac{3}{4}(W(\tfrac{3}{4}) - W(\tfrac{1}{4}))\right] \\ &\quad + b\left[b\int_{\frac{1}{2}}^{\frac{3}{4}} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) - (1 + bW(\tfrac{1}{4}))W(\tfrac{3}{4}) + W(\tfrac{1}{4})(1 + bW(\tfrac{1}{2}))\right]. \end{aligned}$$

Step (iv): $\frac{3}{4} \le \vartheta < 1$:

$$\begin{aligned} z(\vartheta) &= 1 + 4(\vartheta - \tfrac{1}{2}) + 8(\vartheta - \tfrac{3}{4})^2 + \frac{32}{3}(\vartheta - \tfrac{3}{4})^3 \\ &\quad + 4b\left[8(\vartheta - \tfrac{3}{4})\int_{\frac{1}{4}}^{\vartheta - \frac{1}{2}} W(\varsigma - \tfrac{1}{4})\, d\varsigma + 16(\vartheta - \tfrac{3}{4})\int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(u - \tfrac{1}{4})\, du\right. \\ &\quad \left. + 4\int_{\frac{3}{4}}^{\vartheta} W(v - \tfrac{1}{4})\, dv + 8(1 + 2(\vartheta - \tfrac{3}{4}))\int_{\frac{1}{4}}^{\frac{1}{2}} W(\varsigma - \tfrac{1}{4})\, d\varsigma + W(\vartheta)\right]. \end{aligned}$$

We arrange and simplify the above integral with the help of the following formulas [2].

$a) \int_{\frac{3}{4}}^{\vartheta} uW(u-\frac{1}{4})du =$

$\int_{\frac{3}{4}}^{\vartheta} \left(p+\frac{1}{4}\right) W(p)dp = \int_{\frac{3}{4}}^{\vartheta} \left(pW(p) + \frac{1}{4}W(p)\right) dp = \left(\int_{\frac{3}{4}}^{\vartheta} pW(p)dp\right) + \frac{1}{4}\left(\int_{\frac{3}{4}}^{\vartheta} W(p)dp\right)$

$u-\frac{1}{4} = p \;\; ; \;\; u = p+\frac{1}{4} \quad \Rightarrow \quad du = dp.$

$b) \int_{\frac{3}{4}}^{\vartheta} u^2 dW(u) = (u^2W(u))_{\frac{3}{4}}^{\vartheta} - 2\left(\int_{\frac{3}{4}}^{\vartheta} u\, W(u)du\right) = \vartheta^2 W(\vartheta) - \frac{9}{16}W(\frac{3}{4}) - 2\left(\int_{\frac{3}{4}}^{\vartheta} u\, W(u)du\right) \;.$

$u^2 = U \quad \Rightarrow 2u\, du = dU \quad ; \quad dW(u) = dV \;\Rightarrow W(u) = V$

$c) \int_{\frac{3}{4}}^{\vartheta} u\, dW(u) = \vartheta W(\vartheta) - \frac{3}{4}W(\frac{3}{4}) - \left(\int_{\frac{3}{4}}^{\vartheta} W(u)\, du\right) \;\; ;$

$\left(\int_{\frac{3}{4}}^{\vartheta} W(u)dW(u)\right) = \frac{1}{2}\left(W^2(\vartheta) - W^2(\frac{3}{4}) - (\vartheta - \frac{3}{4})\right)$

As a result, the answer $z(\vartheta)$ to the fourth $\vartheta \in [0/75, 1)$ interval is as follows.

$$
\begin{aligned}
z(\vartheta) = {} & 1 + 4(\vartheta - \tfrac{1}{4}) + 8(\vartheta - \tfrac{1}{2})^2 + \frac{32}{3}(\vartheta - \tfrac{3}{4})^3 \\
& + 4b\Bigg[\Big(3 + bW(\tfrac{1}{4})\Big)\int_{\frac{3}{4}}^{\vartheta} W(u)\, du - \Big(b(W(\vartheta) - W(\tfrac{3}{4})) + 4(\vartheta - \tfrac{3}{4})\Big)\int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(\varsigma)\, ds \\
& \quad - \int_{\frac{1}{2}}^{\frac{3}{4}} W(\varsigma)\, ds + \int_{\frac{1}{2}}^{\frac{3}{4}} W(\varsigma - \tfrac{1}{4})\, ds - (2 + bW(\tfrac{1}{4}))\int_{\frac{3}{4}}^{\vartheta} W(u - \tfrac{1}{4})\, du \\
& \quad + \Big(b(W(\vartheta) - W(\tfrac{3}{4})) + 4(\vartheta - \tfrac{3}{4})\Big)\int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(\varsigma - \tfrac{1}{4})\, ds + b\int_{\frac{3}{4}}^{\vartheta} uW(u - \tfrac{1}{4})\, dW(u) \\
& \quad + \vartheta\Big((2 + bW(\tfrac{1}{2}))W(\tfrac{1}{4}) - (2 + bW(\tfrac{1}{4}))W(\vartheta)\Big) + 2\vartheta^2(W(\vartheta) - W(\tfrac{1}{4})) - \frac{7}{8}W(\tfrac{1}{4})\Bigg] \\
& + b^2\Bigg[\Big(b(W(\vartheta) - W(\tfrac{3}{4})) + 4(\vartheta - \tfrac{3}{4})\Big)\int_{\frac{1}{2}}^{\vartheta - \frac{1}{4}} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) \\
& \quad + \int_{\frac{1}{2}}^{\frac{3}{4}} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) - (2 + bW(\tfrac{1}{4}))\int_{\frac{3}{4}}^{\vartheta} W(u - \tfrac{1}{4})\, dW(u) \\
& \quad + \Big(2W(\tfrac{1}{4}) + bW(\tfrac{1}{4})W(\tfrac{1}{2}) + \frac{7}{2}b\Big)W(\vartheta) - W(\tfrac{1}{2})W(\tfrac{1}{4})(2 + bW(\tfrac{3}{4}))\Bigg]
\end{aligned}
$$

As a result, the solution of the stochastic delay differential equation with multiplicative coefficient is obtained as follows. The solution $z(\vartheta)$ is given piecewise as follows.

(i) For $0 \leq \vartheta < \frac{1}{4}$,

$$z(\vartheta) = 0.$$

(ii) For $\frac{1}{4} \leq \vartheta < \frac{2}{4}$,

$$z(\vartheta) = 1 + 4\big(\vartheta - \tfrac{1}{4}\big) + b\big(W(\vartheta) - W(\tfrac{1}{4})\big).$$

(iii) For $\frac{2}{4} \leq \vartheta < \frac{3}{4}$,

$$\begin{aligned} z(\vartheta) &= 1 + 4\big(\vartheta - \tfrac{1}{4}\big) + 8\big(\vartheta - \tfrac{1}{2}\big)^2 \\ &\quad + 4b\left[\int_{\frac{1}{2}}^{\vartheta} \big(W(\varsigma - \tfrac{1}{4}) - W(\varsigma)\big)\, d\varsigma + \vartheta\big(W(\vartheta) - W(\tfrac{1}{4})\big)\right] \\ &\quad + b\left[b\int_{\frac{1}{2}}^{\vartheta} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) - \big(1 + bW(\tfrac{1}{4})\big)W(\vartheta) + W(\tfrac{1}{4})\big(1 + bW(\tfrac{1}{2})\big)\right]. \end{aligned}$$

(iv) For $\frac{3}{4} \leq \vartheta < 1$,

$$\begin{aligned} z(\vartheta) &= 1 + 4\big(\vartheta - \tfrac{1}{4}\big) + 8\big(\vartheta - \tfrac{1}{2}\big)^2 + \tfrac{32}{3}\big(\vartheta - \tfrac{3}{4}\big)^3 \\ &\quad + 4b\left[(3 + bW(\tfrac{1}{4}))\int_{\frac{3}{4}}^{\vartheta} W(u)\, du - (2 + bW(\tfrac{1}{4}))\int_{\frac{3}{4}}^{\vartheta} W(u - \tfrac{1}{4})\, du\right] \\ &\quad + b^2\left[\int_{\frac{1}{2}}^{\frac{3}{4}} W(\varsigma - \tfrac{1}{4})\, dW(\varsigma) - (2 + bW(\tfrac{1}{4}))\int_{\frac{3}{4}}^{\vartheta} W(u - \tfrac{1}{4})\, dW(u)\right]. \end{aligned}$$

The numerical results of the mean and standard deviation with confidence intervals for some different values of ϑ_i in the points $h = \frac{1}{m}$ are shown in tables. The curves in Figure (1) and (2) represent a trajectory of the approximated solution computed by the suggested scheme with a trajectory of the analytic solution.

Table 1: Stochastic time delayed differential with additive noise, $m = 4, k = 1, \sigma = 0.98, b = 0.85$

ϑ_i	$\bar{z}_e$	S_e	LB	UB
0	0	0	0	0
0.25	−0.0043509	0.080249	−0.0040865	−0.0046154
0.5	−0.0084741	0.19613	−0.0079591	−0.0089892
0.75	−0.0097535	0.20931	−0.0091607	−0.010346
1	0.001337	0.11877	0.0012558	0.0014183

Table 2: Stochastic time delayed differential with multiplicative noise, $m = 4, k = 1, \sigma = 0.98, b = 0.85.$

ϑ_i	$\bar{z}_e$	S_e	LB	UB
0	0	0	0	0
0/25	−0.00046861	0.015105	−0.00044013	−0.00049709
0/5	−0.00028699	0.0092508	−0.00026955	−0.00030444
0/75	−0.0010824	0.03489	−0.0010166	−0.0011482
1	0.00019629	0.0063271	0.00018436	0.00020822

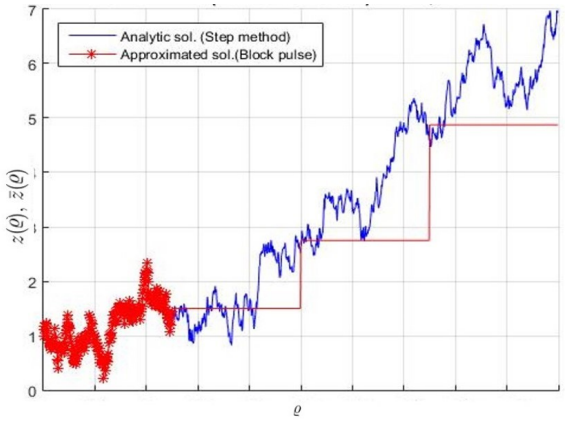


Figure 1. Solving SDDEs(addiditive noise) via BPFs, m=4, k=1.

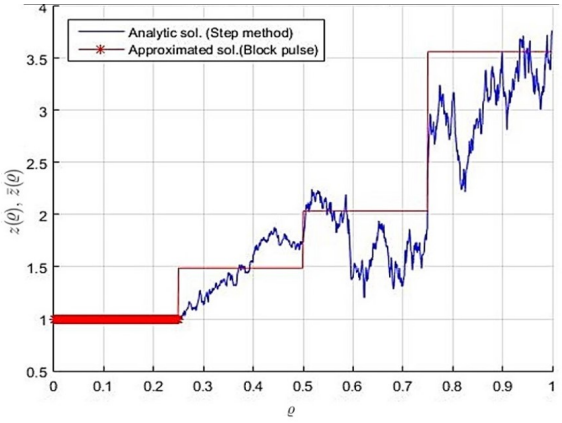


Figure 2 .Solving SDDEs (multiplicative noise) via BPFs, m=4, k=1.

6. **Conclusion**

This paper represents the operational matrices of block pulse functions which have been applied for solving SDDEs with constant time delay. The proposed approach utilizing Basis Functions (BPFs) is advantageous due to its straightforward computational requirements and a commendable degree of accuracy when contrasted with the Euler-Maruyama

method, which operates with a complexity of $O(h^{1/2})$ as noted in Buckwar(2000). Furthermore, this method does not necessitate the use of projection techniques such as Galerkin or collocation methods. The $O(h)$ rate of convergence has ensured the result greatly. Numerical solutions of other orthogonal functions for solving SDDEs is the subject of our subsequent investigation.

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