Shahid Chamran University of Ahvaz

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TOTAL TIME ON TEST TRANSFORM VIA QUANTILE FUNCTION

M. ESFAHANI*, M. AMINI AND G. R. MOHTASHAMI BORZADARAN

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Abstract. This paper investigates the Total Time on Test (TTT) transform through the framework of the quantile function and explores its theoretical properties. The study establishes the connection between the TTT transform and several concepts in reliability theory, including aging and hazard rate functions, expressed in terms of quantile functions. Furthermore, the paper highlights the relationship between the TTT transform and economic indices based on quantile functions, and demonstrates its effectiveness as a tool for measuring income inequality. The main motivation of this study was to establish a unified framework linking reliability theory and income inequality measurement, showing that the TTT transform can not only enhance the analysis of lifetime distributions but also serve as a more sensitive and reliable index of economic inequality. In addition to restating and extending known results, we present new findings that strengthen the role of the TTT transform as both a reliability measure and an economic inequality index. Numerical examples and graphical illustrations are provided to clarify the applicability of the proposed results.

1. Introduction

The quantile function is a fundamental tool in statistics and probability with a central role in characterizing distributions. Entire families of distributions can be introduced through this function, which is also widely used for data modeling and unifying different

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* Department of Statistics, Velayat University, Iranshahr, Iran.

statistical approaches. The use of quantile functions dates back to the 19th century, with early applications in data analysis and distribution modeling by Quetelet [24] and Galton [16]. For a comprehensive study of its properties, applications, and connections with reliability concepts, see [22]. The total time on test (TTT) transform is another important reliability tool with applications in economics, insurance, and aging analysis. First introduced by [2], the properties and applications of the TTT transform have since been studied extensively [18, 22]. One key application is the characterization of distribution functions. Recent works have expanded its scope: [13] introduced new graphical applications of the TTT plot, [6] examined its role in reliability and aging, and [8] employed it as an index of economic inequality, comparing it to established measures. Further extensions for weighted variables were presented in [7, 9]. Several studies, including [23], [10], and [11], have investigated the TTT transform in the bivariate framework and established significant results in this direction. Despite these contributions, a unified framework that simultaneously connects the TTT transform with both reliability concepts and income inequality measures has not been fully developed. The motivation of this study is to fill this gap by reformulating the TTT transform in terms of the quantile function, thereby establishing a coherent structure that enhances its analytical role in reliability theory while also positioning it as a precise and reliable index of income inequality.

The paper is structured as follows. Section 2 provides a detailed overview of the quantile function and the TTT transform, including essential definitions and properties. Section 3 presents the main results, examining the TTT transform in relation to reliability concepts such as aging functions and hazard rates, as well as its connections with economic indicators derived from the quantile function. The section further evaluates the TTT transform as a measure of income inequality and illustrates key findings through numerical examples. Finally, Section 4 offers the study's conclusions.

2. **Preliminaries**

In this section, we introduce the quantile function and its main properties, followed by the definition of the TTT transform. We also review several auxiliary concepts needed for the paper. While the quantile function is defined on real-valued random variables, all results remain valid for lifetime (non-negative) random variables in the context of the TTT transform.

Definition 2.1. Let X be a real-valued continuous random variable with distribution function $F(x)$ (right-continuous). The quantile function of X, denoted by $Q(p)$, is defined as:

$$Q(p) = F^{-1}(p) = \inf\{x : F(x) \geq p\}, \quad 0 \leq p \leq 1. \tag{2.1}$$

For every $x \in \mathbb{R}$ and $0 < p < 1$ we have $F(x) \geq p$ if and only if $Q(p) \leq x$.

If there exists X such that $F(x) = p$, then $F(Q(p)) = p$ and $Q(p)$ is the smallest value of x. When $F(x)$ is continuous and strictly increasing, $Q(p)$ is uniquely determined by solving $F(x) = p$. If $f(x)$ is the density of X, then $f(Q(p))$ is the density quantile function. The derivative of $Q(p)$, denoted $q(p) = Q^{'}(p)$, is called the quantile density function. Differentiating $F(Q(p)) = p$ gives the identity $q(p)f(Q(p)) = 1$. For further details on properties and applications of the quantile function, see [22]. In what follows, we introduce the definition and characteristics of the TTT transform. Since the TTT transform is also used to compare random variables and their distributions, a stochastic order can be defined. Let X and Y be random variables with distribution functions $F(\cdot)$ and $G(\cdot)$, and densities $f(\cdot)$ and $g(\cdot)$, respectively. Define the survival functions $\bar{F}(\cdot) = 1-F(\cdot)$ and $\big(\bar{G}(\cdot) = 1 - G(\cdot)\big)$, and let $Q_X(p)$ and $Q_Y(p)$ denote the corresponding quantile functions.

Definition 2.2. For a lifetime continuous random variable X, the TTT transform is defined as:

$$T_X(p) = \int_0^{Q(p)} \bar{F}(x)dx, \quad p \in (0,1), \tag{2.2}$$

It follows that $T_X(1)$ equals the expectation of X, which may be finite or infinite. For finite mean $\mu = E(X)$, the scaled TTT transform for $p \in (0,1)$ is $\varphi_X(p) = \frac{T_X(p)}{\mu}$.

If the distribution function is used in place of the survival function in (2.2), one obtains the location independent riskier (LIR) transform,

$$LIR_X(p) = \int_0^{Q_X(p)} F(x), dx, \quad p \in (0,1),$$

with the identity $T_X(p) + LIR_X(p) = Q(p)$.

Definition 2.3. Let X and Y be random variables with distribution functions $F(\cdot)$ and $G(\cdot)$, both with finite means. For $p \in (0,1)$:

a) X is smaller than Y in the TTT transform order, denoted $X \leq_{ttt} Y$, if $T_X(p) \leq T_Y(p)$;

b) X is smaller than Y in the LIR transform order, denoted $X \leq_{lir} Y$, if $LIR_X(p) \leq LIR_Y(p)$.

The TTT transform order was first introduced in [18] to compare nonnegative random variables, with further applications discussed there. The LIR order and its properties were studied in [14]. Throughout this paper, we adopt the notations of [25], noting that stochastic orders describe relationships between probability distributions.

Definition 2.4. Let X and Y be random variables with distribution functions $F(\cdot)$ and $G(\cdot)$.

- X is stochastically smaller than Y ($X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x.
- X is smaller in the hazard rate order than Y ($X \leq_{hr} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing.
- X is smaller in the convex transform order than Y ($X \leq_c Y$) if $Q_Y(F(x))$ be convex.

It follows that $X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{ttt} Y$. Conditions for the converse relations are given in [25].

Definition 2.5. Let $F(\cdot)$ be a lifetime distribution.

- $F(\cdot)$ is $IFR(DFR)$[1] if its hazard rate $r(x) = \frac{f(x)}{\bar{F}(x)}$ is increasing (decreasing).
- $F(\cdot)$ is $NBU(NWU)$[2] if $\bar{F}(x+y) \leq (\geq) \bar{F}(x)\bar{F}(y)$ for $x, y \geq 0$.

Definition 2.6. Let X be a lifetime random variable with distribution $F(\cdot)$ and quantile function $Q(p)$. In terms of $Q(p)$, several reliability-related quantile functions are defined as follows:

- Hazard quantile: $H(u) = [(1-u)q(u)]^{-1}$.
- Mean residual quantile: $M(u) = (1-u)^{-1} \int_u^1 (1-p)q(p)dp$.
- Residual variance quantile: $V(u) = (1-u)^{-1} \int_u^1 M^2(p)dp$.
- Reversed hazard quantile: $\Lambda(u) = [uq(u)]^{-1}$.
- Reversed mean residual quantile: $R(u) = u^{-1} \int_0^u pq(p)dp$.
- Reversed variance residual quantile: $D(u) = u^{-1} \int_0^u R^2(p)dp$.

[1]Increasing(decrasing) failure rate

[2]New better(worse) than used

Definition 2.7. Let $F(\cdot)$ be a lifetime distribution with density $f(\cdot)$ and survival function $\bar{F}(\cdot)$. Denote by μ the mean and by μ^r the $r-$th central moment of $F(\cdot)$. Several important aging and reliability classes are characterized as follows:

a) $F \in IFR \Longleftrightarrow \frac{f(x)}{\bar{F}(x)} \nearrow x$

b) $F \in IFRA \Longleftrightarrow -\frac{\log \bar{F}(x)}{x} \nearrow x > 0$

c) $F \in NWUE \Longrightarrow \bar{F}(t) \leq \frac{\mu}{\mu+t}, \quad t \geq 0$

d) $F \in NBUE \Longrightarrow \bar{F}(t) \geq \begin{cases} 1-\frac{t}{\mu}, & t \leq \mu \\ 0, & t > \mu \end{cases}, \quad t \geq 0$

e) $F \in NBRU(NWRU) \Longleftrightarrow \int_{x+t}^{\infty} \frac{\bar{F}(u)du}{\bar{F}(x)} \leq (\geq) \int_x^{\infty} \bar{F}(u)du, \quad t \geq 0$

f) $F \in RNBU(RNWU) \Longleftrightarrow \frac{\bar{F}(x+t)}{\bar{F}(t)} \leq (\geq) \frac{1}{\mu} \int_x^{\infty} \bar{F}(u)du, \quad t \geq 0$

g) $F \in RNBUE(RNWUE) \Longleftrightarrow 2\mu \int_x^{\infty} \bar{F}(u)du \leq (\geq) \mu_2^{'} \bar{F}(x)$

h) $F \in RNBRU(RNWRU) \Longleftrightarrow \frac{(\int_x^{\infty} \bar{F}(u)du)(\int_t^{\infty} \bar{F}(u)du)}{\int_{x+t}^{\infty} \bar{F}(u)} \geq (\leq)\mu$

i) $F \in RNBRUE(RNWRUE) \Longleftrightarrow \frac{\int_x^{\infty}\int_t^{\infty} \bar{F}(u)dudt}{\int_x^{\infty} \bar{F}(u)du} \leq (\geq) \frac{\mu_2^{'}}{2\mu}$

j) $F \in HRNBUE(HRNWUE) \Longleftrightarrow \bar{F}_Z(t) \leq (\geq) e^{-\frac{t}{\mu_Z}}$

k) $F \in GIMRL(GDMRL) \Longleftrightarrow \frac{\int_t^{\infty} \bar{F}(u)du}{\bar{F}(t+x)} \nearrow (\searrow) t$

l) $F \in NBUR_h \Longleftrightarrow \frac{F(x)}{\int_0^x \bar{F}(t)}dt \nearrow x$

m) $F \in NBEE \Longleftrightarrow \int_0^{\infty} \int_t^{\infty} \bar{F}(x)dxdt \leq \mu^2$

n) $F \in NBUC \Longleftrightarrow \int_{x+y}^{\infty} \bar{F}(t)dt \leq \bar{F}(x) \int_y^{\infty} \bar{F}(t)dt, \quad \forall x, y \geq 0$

o) $F \in NBUCA \Longleftrightarrow \frac{\int_0^{\infty}\int_x^{\infty} \bar{F}(u+t)dudx}{\int_0^{\infty}\int_x^{\infty} \bar{F}(u)dudx} \leq \bar{F}(t), \quad t > 0$

p) $F \in HNBUC \Longrightarrow \frac{\int_t^{\infty} \bar{F}(x+t)dxdt}{\int_t^{\infty} \bar{F}(x)dx} \leq \bar{F}(t), \qquad t \geq 0$

q) $F \in NBELC \Longrightarrow \frac{\int_x^{\infty}\int_y^{\infty} \bar{F}(u)dudy}{\int_x^{\infty} \bar{F}(y)dy} \leq \mu, \qquad x \geq 0$

For further details, see [22].

3. Main Results

In this section we express the TTT transform in terms of the quantile function and derive relations to reliability concepts. Assume $F(x)$ is continuous and strictly increasing so that $Q(p)$ is well-defined. Putting $x = Q(p)$ in the definition of $T_X(p)$ yields

$$T_X(p) = \int_0^{Q(p)} \bar{F}(x), dx = \int_0^p (1-u)q(u), du, \quad p \in (0,1). \tag{3.1}$$

Thus $T_X^{'}(p) = (1-p)q(p)$ and $Q(p) = \int_0^p \frac{T^{'}(u)}{1-u}du$. Table 1 summarizes relations between T_X and several quantile-based reliability functions.

TABLE 1. Relations between TTT transform and reliability concepts

Reliability Concept	Relation
$H(p)$	$T(p) = \int_0^p H(u)du$
$M(p)$	$T(p) = \mu - (1-p)M(p)$
$V(u)$	$T(p) = \mu - (1-p)[(1-p)V^{'}(p) - V(p)]^{\frac{1}{2}}$
$\Lambda(p)$	$T(p) = Q(p) - \int_0^p \frac{1}{\Lambda(u)}du$
$R(p)$	$T(p) = Q(p) - pR(p)$
$D(p)$	$T(p) = Q(p) - p^3D(p)$

3.1. **Percentile Residual Life Function.** In this subsection, we present results on the relationship between the percentile residual life function and the TTT transform within the framework of the quantile function. Illustrative examples are also provided for clarity. When the use of conventional measures such as the mean and variance is limited, alternative statistical tools are needed to compare random variables. The same applies to residual life functions. In situations involving censored data or highly skewed distributions, where empirical counterparts of the mean and variance of residual life are unreliable, the percentile residual life function serves as an appropriate alternative. This function was first introduced by [17], and further important properties and applications were discussed by [19]. For any $0 < \alpha < 1$, the α-th percentile residual life function is defined as the α-th percentile of the residual life distribution of X. This function, denoted by $p_\alpha(x)$, is given by

$$p_\alpha(x) = Q\big(1 - (1-\alpha)\bar{F}(x)\big) - x. \tag{3.2}$$

As shown in [22] through two illustrative examples, the percentile residual life function does not uniquely determine the underlying distribution $F(x)$. For $x = Q(p)$, the corresponding α-th percentile residual quantile function can be obtained from (3.2) as

$$P_\alpha(p) = p_\alpha(Q(p)) = Q\big(1 - (1-\alpha)(1-p)\big) - Q(p). \tag{3.3}$$

It was later established by [21] that the quantile function $Q(p)$ can be uniquely determined from $P_\alpha(p)$ together with its relationship to the quantile function on a given interval. Using the connection between the TTT transform and the quantile function, $p_\alpha(x)$ and $P_\alpha(p)$

can be expressed as

$$p_\alpha(x) = \int_0^{1-(1-\alpha)\bar{F}(x)} \frac{T'(u)}{1-u} du - x, \tag{3.4}$$

$$P_\alpha(p) = \int_0^{1-(1-\alpha)(1-p)} \frac{T'(u)}{1-u} du - \int_0^{p} \frac{T'(u)}{1-u} du = \int_p^{p+\alpha(1-p)} \frac{T'(u)}{1-u} du. \tag{3.5}$$

For $\alpha \in (0,1)$, a random variable X is said to have an increasing percentile residual life, denoted $IPRL(\alpha)$, if $P_\alpha(p)$ is increasing in $p \in (0,1)$.

Theorem 3.1. *Let F be a continuous distribution with a quantile function $Q(p)$ that is differentiable almost everywhere on $[0,1)$. Then F belongs to the $IPRL(\alpha)$ class if and only if $l(p)$ is increasing on $[0,1)$, where $l(p) = \frac{\partial}{\partial p}[LIR(p)]$.*

Proof. From equation (3.6), we conclude that $F \in IPRL(\alpha)$ if and only if

$$Q(\alpha + (1-\alpha)p) - Q(p) \qquad is \quad increasing \quad in \quad p \in (0,1);$$

which is equivalent to $(1-\alpha)q(\alpha+(1-\alpha)p) \geq q(p)$ a.e. on $[0,1)$. By the definition of the location independent riskier (LIR) transform, we have $q(p) = \frac{l(p)}{p}$ where $l(p) = \frac{\partial}{\partial p}[LIR(p)]$. Thus, the condition becomes

$$\frac{(1-\alpha)l(\alpha+(1-\alpha)p)}{\alpha+(1-\alpha)p} \geq \frac{l(p)}{p},$$

a.e. on $[0,1)$; Rearranging yields

$$l(\alpha+(1-\alpha)p) \geq \frac{\alpha+(1-\alpha)p}{(1-\alpha)p} l(p) \geq l(p),$$

a.e. on $[0,1)$; Hence $F \in IPRL(\alpha)$ if and only if $l(p)$ is increasing on $[0,1)$. □

Let $0 < \alpha < 1$ and X and Y be two random variables with percentile residual functions $p_{X,\alpha}(x)$ and $p_{Y,\alpha}(x)$ respectively. If $p_{X,\alpha}(x) \leq p_{Y,\alpha}(x)$ for all x, then X is said to be smaller than Y in the α-percentile residual life order, denoted by $X \leq_{\alpha-rl} Y$.

Proposition 3.2. *If $X \leq_{\alpha-rl} Y$ for all $\alpha \in (0,1)$, then $X \leq_{ttt} Y$.*

Proof. [15] established that for all $\alpha \in (0,1)$,

$$X \leq_{\alpha-rl} Y \Longleftrightarrow X \leq_{hr} Y.$$

Since the hazard-rate order implies the TTT transform order, it follows directly that

$$X \leq_{\alpha-rl} Y \Longrightarrow X \leq_{ttt} Y.$$

□

Remark 3.3. For any $t \leq t^{'}$ it follows that if $X + t \leq_{\alpha-rl} Y + t^{'}$ then $X + t \leq_{ttt} Y + t^{'}$.

Suppose $H(x) = T^{-1}(x)$ is the invers of the TTT transform. As shown in [18], $H(x)$ is the distribution function of the random variable $X_{ttt} = T(F(X))$ which is referred to as the observed total time on test when X is observed. Differentiating $H(x)$, the corresponding density function is $h(x) = \frac{\partial}{\partial x}H(x) = \frac{1}{t(T^{-1}(x))}, \quad x \in (0, E(X))$, where $t = \frac{\partial}{\partial x}T(x)$ is the TTT density as defiend in (4).

Now define a new random variable $X_{lir} = Lir(F(X))$ called the observed location independent riskier when X is observed. Its distribution function is given by

$$J(y) = P(X_{lir} \leq y) = P(Lir(F(X)) \leq y) = P(F(X) \leq Lir^{-1}(y)).$$

Since $U \overset{st}{=} F(X)$, we obtain $J(y) = P(U \leq Lir^{-1}(y)) = Lir^{-1}(y)$. Thus, the inverse of the LIR transform is the distribution function of the random variable X_{lir}. According to Theorem 3.5 of [12], the following result holds.

Proposition 3.4. *For $\alpha \in (0, 1)$, an absolutely continuous random variable X belongs to the $IPRL(\alpha)$ class if and only if,*

$$j(x) \geq j(Lir^{-1}(\alpha + (1 - \alpha)Lir(x)))), \qquad x \in (0, E[X]).$$

Proof. Analogous to the observed total time on test, for the observed location independent riskier we can write $l(p) = \frac{1}{j(LIR(p))}, \quad p \in (0, 1)$. From Theorem 3.1, it follows that $X \in IPRL(\alpha)$ if and only if,

$$j(LIR(p)) \geq j(LIR(\alpha + (1 - \alpha)p)), \quad p \in (0, 1),$$

Equivalently, setting $x = LIR(p)$, this condition becomes

$$j(x) \geq j(LIR(\alpha + (1 - \alpha)Lir^{-1}(x))), \quad x \in (0, E[X])),$$

which is the desired result. □

3.2. **Agieng Concepts.** In this section we investigate ageing properties of lifetime distributions using the quantile function, with emphasis on the role of the TTT and scaled TTT transforms. While several results have been previously established (see [22]), we reformulate them in the quantile–TTT framework and extend them to obtain new characterizations and bounds. [1] showed that a necessary and sufficient condition for a distribution X to be $DMTTF$ (resp. $IMTTF$) is that $\frac{\varphi(p)}{p}$ is decreasing (resp. increasing). Using the results presented in [1], this condition can be expressed as

$$p(1-p)q(p) \leq (\geq) T_X(p), \quad p \in (0,1). \tag{3.6}$$

Similarly, based on [3], we have

$$X \in NBUE(NWUE) \Longleftrightarrow \int_0^p (1-u)q(u)du \geq (\leq) p\mu.$$

From the definition of $HNBUE$ and equation (3.6), additional ageing classes can be described:

- X is HNBUE (resp. HNWUE) iff

$$\int_0^p (1-u)q(u)du \leq (\geq)\mu\Big(1-\exp\Big[-\tfrac{Q(p)}{\mu}\Big]\Big), \quad 0 \leq p \leq 1. \tag{3.7}$$

- X is DMRL (resp. IMRL) iff

$$(1-p)^2 q(p) + T_X(p) \geq (\leq)\mu. \tag{3.8}$$

Remark 3.5. Using results from [22], the following conditions can be expressed in terms of the quantile–TTT relationship:

- X is $NBUHR$ if and only if $(1-p)q(p) \leq (\geq) t(0)$;
- X is $NBUFHA$ if and only if $-\frac{\log(1-p)}{\int_0^p \frac{t(u)}{1-u}du} \leq (\geq) t(0)$;

These reformulations highlight the novelty of our approach: classical ageing concepts are systematically re-expressed via the quantile-based TTT transform, yielding sharper inequalities and making the underlying structure more transparent. Moreover, in cases where the TTT transform cannot be obtained in closed form and requires complex or numerical calculations, our results provide useful analytic bounds that extend its applicability.

Proposition 3.6. *Let X be a lifetime random variable with distribution function F. Then*

(a) $F \in NBUE \Longrightarrow \varphi_X(p) \geq \frac{1}{2} I_{[\mu \geq Q(p)]} + \left(\frac{Q(p)}{\mu} - \frac{Q^2(p)}{2\mu^3} \right) I_{[\mu > Q(p)]}$,

(b) $F \in NWUE \Longrightarrow \varphi_X(p) \leq \ln(\mu + Q(p)) - \ln(\mu), \quad p \in (0,1)$.

Proof. **(a)** Integrating both sides of relation (d) in Definition 2.7 over the interval $(0, Q(p))$ yields the stated inequality.

(b) Similarly, applying relation (c) of Definition 2.7 immediately gives the second result. □

Proposition 3.7. *Let $F(\cdot)$ be a lifetime distribution. Then, for all $p, w, z \in (0,1)$,*

(a) $F \in NBRU(NWRU) \Leftrightarrow \frac{\mu - \int_0^{Q(p)+t} \bar{F}(u)du}{\bar{F}(x)\mu} \leq (\geq) 1 - \varphi_X(p)$

(b) $F \in RNBU(RNWU) \Longleftrightarrow \varphi_X(p) \leq (\geq) 1 - \frac{\bar{F}(Q(p)+t)}{\bar{F}(t)}$

(c) $F \in RNBUE(RNWUE) \Longleftrightarrow \varphi_X(p) \geq (\leq) 1 - \frac{\mu_2'}{2\mu^2} \bar{F}(t)$

(d) $F \in RNBRU(RNWRU) \Longleftrightarrow 1 - \varphi_X(z) \leq (\geq)(1 - \varphi_X(p))(1 - \varphi_X(w))$,

$Q(z) = Q(p) + Q(w)$

(e) $F \in RNBRUE(RNWRUE) \Longleftrightarrow \int_{Q(w)}^{\infty} (1 - \varphi_X(p)) q(p) dp \leq (\geq) \frac{\mu_2'(1 - \varphi_X(p))}{2\mu}$,

(f) $F \in HRNBUE(HRNWUE) \Longleftrightarrow \varphi_Z(p) \leq (\geq) 1 - \exp(-\frac{Q_Z(p)}{\mu_Z})$

(g) $F \in GIMRL(GDMRL) \Longleftrightarrow \frac{\mu(1 - \varphi_X(p))}{\bar{F}(Q(p)+x)} \nearrow (\searrow) p$

Proof. All parts follow by substituting $Q(p)$(or $F(x) = p$) into the corresponding integral characterizations in Definition 2.7 and then rewriting the resulting inequalities in terms of the TTT transform $T_X(p)$ and its scaled version $\varphi_X(p)$. We give details for items (a), (e) and (g); the remaining cases are analogous.

To prove each of the cases, it is enough to use Definition 2.7 We prove items (a), (e) and (g). Other cases can be proved by a similar argument.

(a) Using relation (e) of Definition 2.7, set $F(x) = p$ and integrate over $(0, Q(p) + t)$. Rearranging the resulting inequality and recalling

$$\frac{\mu - \int_0^{Q(p)+t} \bar{F}(u)du}{\mu \bar{F}(x)} \leq (\geq) 1 - \varphi_X(p),$$

which is the claimed equivalence for $NBRU$ (resp. $NWRU$).

(e) For the $RNBRUE$ (resp. $RNWRUE$) class use its defining inequality and set $Q(w) = x$. Noting that $\varphi_F(p) = \dfrac{(1-p)q(p)}{\mu}$ and substituting $Q(p) = t$ into the integrals, a straightforward simplification gives $\frac{\int_x^\infty \frac{(1-\varphi_F(p))}{f(Q(p))}dp}{\int_x^\infty \bar{F}(u)du} \leq (\geq)\frac{\mu_2'}{2\mu}$, which is the stated equivalence

(g) Finally, for the $GIMRL$ (resp. $GDMRL$) class substitute $Q(p) = t$ into relation (k) of Definition 2.7 and express the left-hand side by using

$$1 - \varphi_X(p) = 1 - \frac{1}{\mu}\int_0^{Q(p)} \bar{F}(u), du.$$

After simplification one obtains the monotonicity condition $\frac{\mu(1-\varphi_X(p))}{\bar{F}(Q(p)+x)} \nearrow (\searrow)p$, which completes the derivation for (g). The remaining equivalences listed in the proposition follow from analogous substitutions and algebraic rearrangements of the corresponding items in Definition 2.7. □

Proposition 3.8. *Let $F(\cdot)$ be a lifetime distribution. For all $p \in (0,1)$ the following hold:*

(a) $F \in NBUR_h \iff F \in IFRA$.

(b) $F \in NBEE \iff \int_0^1 \dfrac{(1-\varphi_X(p))q(p)}{\mu}dp \leq 1$.

Proof. **(a)** By part (l) of Definition 2.7 we have the characterization

$$F \in NBUR_h \iff \frac{F(x)}{\int_0^x \bar{F}(t)dt} \text{ is increasing in } x.$$

Setting $x = Q(p)$ and using $F(Q(p)) = p$ together with

$$\int_0^{Q(p)} \bar{F}(t)dt = T_X(p) = \mu\varphi_X(p),$$

we obtain

$$\frac{F(Q(p))}{\int_0^{Q(p)} \bar{F}(t)dt} = \frac{p}{\mu\varphi_X(p)}.$$

Thus the monotonicity of $\frac{F(x)}{\int_0^x \bar{F}(t)dt}$ in x is equivalent to the monotonicity of $\frac{p}{(\mu\varphi_X(p))}$ in p. By standard equivalences (see [22]), this monotonicity is exactly the $IFRA$ property. Hence $F \in NBUR_h$ if and only if $F \in IFRA$.

(b) Item (m) of Definition 2.7 states that

$$F \in NBEE \iff \int_0^\infty \int_t^\infty \bar{F}(x)dxdt \leq \mu^2.$$

TABLE 2. Ageing-class membership of Weibull distributions with $\lambda = 1$ and various k

Shape k	Mean μ	DMTTF	NBUE	HNBUE	DMRL
0.7	1.298	No	No	No	No
1.0	1.000	Boundary	Boundary	Boundary	Boundary
1.5	0.903	Yes	Yes	Yes	Yes
2.0	0.886	Yes	Yes	Yes	Yes

Change variables using $x = Q(p)$ and $dx = q(p)dp$. Observing that

$$\int_t^\infty \bar{F}(x), dx = \mu\big(1 - \varphi_X(p_t)\big)$$

for the appropriate p_t satisfying $Q(p_t) = t$, one obtains after rearrangement the equivalent condition

$$\int_0^1 \frac{(1 - \varphi_X(p))q(p)}{\mu} dp \leq 1.$$

Therefore $F \in NBEE$ if and only if the stated integral inequality holds. □

Illustrative Example: Weibull Distribution and Ageing Properties

To illustrate the applicability of the proposed characterizations, we consider the Weibull distribution with scale parameter $\lambda = 1$ and various shape parameters k. The Weibull family is widely used in reliability analysis because it encompasses both decreasing and increasing failure rate behaviours depending on the value of k. By performing computations for this distribution, we obtain $q(p) = \frac{1}{k}\frac{\left(-\ln(1-p)\right)^{1/k-1}}{1-p}$, and $T_X(p) = \frac{1}{k}\gamma\left(\frac{1}{k}, -\ln(1-p)\right)$, where $\gamma(\cdot,\cdot)$ denotes the lower incomplete gamma function. We numerically evaluated $T_X(p)$ and the associated inequalities defining several ageing classes, namely: DMTTF, NBUE, HNBUE and DMRL, for selected values of k. Table 2 summarizes whether the corresponding inequality holds for all $p \in (0, 1)$. To provide more concrete insight, Table 3 reports numerical values of the quantile $Q(p)$, the TTT transform, and the benchmark quantities appearing in the inequalities for three representative probabilities $p = 0.2, 0.5, 0.8$ with $k = 1.5$. The results confirm that the Weibull distribution with shape $k > 1$ indeed belongs to the classes NBUE, HNBUE and DMRL.

TABLE 3. Numerical evaluation for Weibull distribution with $k = 1.5,\ \lambda = 1$.

p	$Q(p)$	$T_X(p)$	$p\mu$	$\mu(1 - e^{-Q(p)/\mu})$	$(1-p)^2 q(p) + T_X(p)$
0.2	0.303	0.198	0.181	0.199	0.928
0.5	0.766	0.451	0.452	0.460	0.931
0.8	1.741	0.775	0.722	0.766	0.921

These computations highlight the flexibility of our characterizations: by adjusting the shape parameter k of the Weibull distribution, one can move across different ageing classes, thereby demonstrating the discriminative power of the proposed criteria.

The TTT transform is intrinsically connected to the notion of convexity, making it a powerful tool for defining and studying convex orders. One of its important applications lies in characterizing distributional properties through convexity-based comparisons. Building on this idea, [22] introduced several convexity classes, formulated their defining conditions for lifetime distributions, and explored their properties and applications. In what follows, we restate the conditions for these convexity classes in terms of the TTT transform. Each characterization can be derived by substituting $F(x) = p$ and $F(y) = z$ into the corresponding definitions, thereby providing a quantile–TTT representation that unifies classical convexity concepts within a reliability framework.

Proposition 3.9. *If $F(\cdot)$ be a lifetime distribution. Then, for all $p, z \in (0,1)$:*

(a) $F \in NBUC \iff 1 - \frac{1}{\mu}\int_0^{Q(p)+Q(z)} \bar{F}(t)dt \le (1-p)(1-\varphi_X(p))$,

(b) $F \in NBUCA \iff \frac{\int_0^1 [1 - \frac{1}{\mu}\int_0^{Q(p)+Q(z)} \bar{F}(u)du]q(p)dp}{\int_0^1 (1-\varphi_X(p))q(p)dp} \le 1 - z$,

(c) $F \in HNBUC \iff 1 - \frac{1}{\mu}\int_0^{2Q(p)} \bar{F}(x)dx \le (1-p)(1-\varphi_X(p))$,

(d) $F \in NBELC \iff \int_{Q(z)}^{\infty} \frac{(1-\varphi_X(p))q(p)}{\mu} dp \le 1 - \varphi_X(z)$.

Remark 3.10. [20] introduced and studied a family of univariate stochastic orders parameterized by a function $h(\cdot)$. They defied the generalized total time on test ($GTTT$) transform as follow:

For $h \in \Lambda$, where $\Lambda = \{h(u) > 0, \forall u \in (0,1), h(u) = 0, \forall u \notin (0,1)\}$ the $GTTT$ transform associated with distribution F is

$$T_{hF}(p) = \int_{-\infty}^{Q(p)} h(F(u))du, \quad p \in (0,1). \tag{3.9}$$

Furthermore, if

$$\int_{-\infty}^{Q_X(p)} h(F(u))du \leq \int_{-\infty}^{Q_Y(p)} h(G(u))du, \quad p \in (0,1), \tag{3.10}$$

then X is said to smaller than Y in the generalized total time on test ($GTTT$) transform order, denoted by $X \leq_{ttt}^{(h)} Y$.

It is known that if $X \leq_c Y$ (convex order),

$$\frac{\partial}{\partial x} Q_Y(F(x)) = \frac{f(x)}{g[Q_Y(F(x))]}$$

is non-increasing in x, for $p \in (0,1)$. Now, if we set $h(u) = g(Q_Y(u))$ in definition of $GTTT$ transform and differentiate with respect to p, we obtain

$$\frac{\partial}{\partial p} T_{hF}(p) = \frac{g[Q_Y(p)]}{f[Q_X(p)]}, \tag{3.11}$$

Thus, the non-increasing (resp. non-decreasing) behavior of equation (3.11) implies that $T_{hF}(p)$ is concave (resp. convex). Therefore, $X \leq_c Y$ if and only if $T_{hF}(p)$ be concave.

This provides a new functional perspective on convex order via the generalized TTT transform.

3.3. **Economic Indices.** A central factor in social stability is the fair distribution of income, as unequal allocation can lead to class divisions and social problems. For this reason, measuring inequality has long been studied by economists and policymakers. Several indices have been proposed, including Lorenz, Gini, Pietra, Bonferroni, Zenga, and Leimkuhler (see [4]). Let income be a non-negative random variable with distribution $F(\cdot)$ and mean μ. The Lorenz curve, $L(p)$, is defined as

$$L(p) = \frac{1}{\mu} \int_0^p Q(u)du, \quad 0 \leq p \leq 1. \tag{3.12}$$

and is convex, continuous, and increasing in $[0,1]$, with $L(0) = 0$, $L(1) = 1$, and $L(p) \leq p$. Perfect equality corresponds to the diagonal of the unit square. The curve is sensitive to additive changes in income but invariant to proportional scaling. From the Lorenz transform, several well-known indices are derived, such as $G = 1 - 2\int_0^1 L(u)du$ and $P = F(\mu) - L(F(\mu))$, together with the Leimkuhler curve $K(p) = 1 - L(1-p)$, and the Bonferroni curve $B(p) = \frac{L(p)}{p}$, with coefficient $B = 1 - \int_0^1 B(u)du$. The Zenga index is defined as $Z(p) = 1 - \frac{(1-p)L(p)}{p(1-L(p))}, \quad 0 \leq p \leq 1.$

TABLE 4. Relation with economic indices

Index	Relation
Lorenz	$L(p) = \varphi(p) - \frac{(1-p)Q(p)}{\mu}$
Bonfferoni	$B(p) = \frac{\varphi(p)}{p} - \frac{(1-p)Q(p)}{p\mu}$
Zenga	$Z(p) = 1 - \frac{1-p}{p}\frac{\mu\varphi(p)-(1-p)Q(p)}{\mu-[\mu\varphi(p)-(1-p)Q(p)]}$
Leimkuhler	$K(p) = 1 - \varphi(1-p) + \frac{pQ(1-p)}{\mu}$
Gini	$G = 1 - 2\int_0^1[\varphi(u) - \frac{(1-u)Q(u)}{\mu}]du$
Pietra	$P = 1 - \varphi(F(\mu))$

In addition to these, the Total Time on Test transform, widely used in reliability, has been shown to capture inequality characteristics ([5], [7]). Table 4 summarizes how classical indices can be expressed in terms of the TTT transform. This unified representation demonstrates the close relationship between the TTT framework and traditional measures, providing a broader basis for using the TTT plot as an inequality index. The TTT plot itself is an increasing curve starting at the origin and ending at point $(1, 1)$. Like the Lorenz curve, it is typically convex, though it may also be concave. Its main strength is its intuitive interpretation and ability to distinguish distributions that are otherwise difficult to separate. A convex TTT plot, like the Lorenz curve, reflects higher inequality with greater curvature, whereas a concave curve above the diagonal suggests lower inequality. Overall, the TTT plot satisfies the key properties expected from a valid inequality index—convexity, sensitivity to additive shifts, and invariance under proportional scaling—making it a powerful complement to existing measures. Numerical examples further illustrate these properties.

Proposition 3.11. *Let X be a non-negative random variable with scale TTT transform φ_X and finite mean μ, representing the income of individuals in a society.*

(a) *If $Y = X + a$, with $a > 0$, then φ is not invariant with respect to Y. In other words, the TTT transform is sensitive to the addition of a fixed amount to all incomes in the society. This implies that adding the same positive value to everyone's income reduces the measured level of inequality.*

(b) *If $Z = cX$, with $c > 0$, then φ is invariant with respect to Z. In other words, the TTT transform is unaffected by proportional changes in all incomes. This means*

that multiplying all incomes by the same positive constant leaves the degree of inequality unchanged.

Proof. **(a)** Since $T_Y(p) = T_X(p) + a$ and $\mu_Y = \mu_X + a$, it follows that

$$\varphi_Y(p) = \frac{T_Y(p)}{\mu_Y} = \frac{T_X(p) + a}{\mu_X + a} = \frac{\mu}{\mu + a}, \varphi_X(p) + \frac{a}{\mu + a}.$$

This expression shows that the TTT transform changes under the addition of a fixed constant $a > 0$, thus confirming its sensitivity.

(b) Since $T_Z(p) = cT_X(p)$ and $\mu_Z = c\mu_X$, we obtain

$$\varphi_Z(p) = \frac{T_Z(p)}{\mu_Z} = \frac{cT_X(p)}{c\mu_X} = \varphi_X(p).$$

Hence, the TTT transform remains unchanged under proportional scaling of incomes, establishing its invariance in this case. □

3.4. **Illustrative Examples.** This section presents numerical examples that showcase the practical use of the ideas discussed earlier in the paper. The accompanying plots provide a visual perspective, highlighting characteristic patterns and differences among various lifetime distributions. These illustrations serve to make the theoretical concepts more tangible and intuitive, allowing the reader to grasp their implications without delving into technical details.

Example 5.1. To compare the effectiveness of Lorenz and TTT curves in detecting economic inequality, we assume that the income data in four societies follow Weibull distributions with scale parameter 3 and shape parameters $\alpha_i, i = 1, 2, 3, 4$ equal to $0.9, 0.91, 0.92, 0.93$, respectively. Figures 1 and 2 display the TTT and Lorenz curves for these distributions. In Figure 1, the TTT curves are convex and lie below the diagonal; their convexity decreases as the shape parameter α_i increases, clearly distinguishing the distributions. In Figure 2, the Lorenz curves are also convex and below the diagonal but almost overlap, making it difficult to differentiate the distributions.

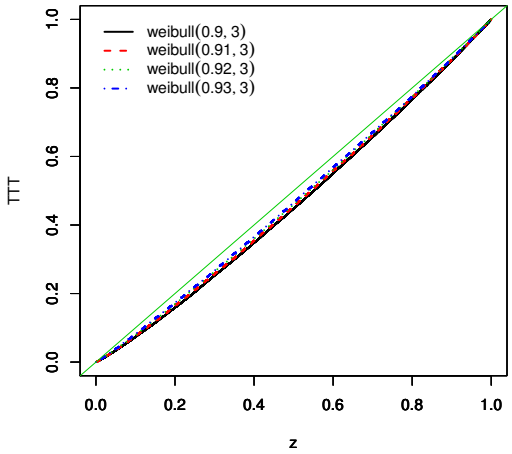


FIGURE 1. TTT plots for Weibull data sets.

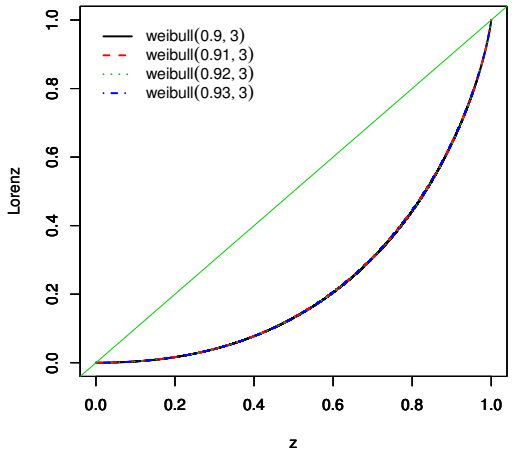


FIGURE 2. Lorenz curves for Weibull data sets.

Example 5.2. In this example, we examine data on household incomes in Iran for the years 1390 to 1399 and compare income inequality across these years by plotting the TTT and Lorenz curves. These curves are presented in Figures 3 and 4, respectively. As observed, comparing income distributions using the Lorenz curve can be somewhat challenging, whereas the TTT curve allows for a more straightforward comparison. It is noteworthy that the TTT curve in this example is concave, which is a consequence of

the increasing hazard rate function of the studied data. Additionally, in concave TTT plots, the further the curve is from the diagonal of the unit square, the lower the income inequality in the corresponding society.

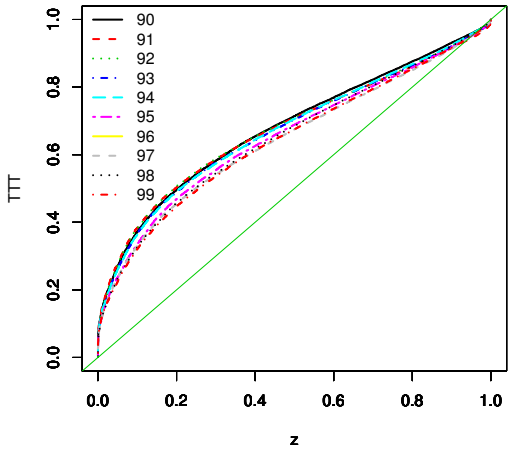


FIGURE 3. TTT plots for income data from 1390-1399.

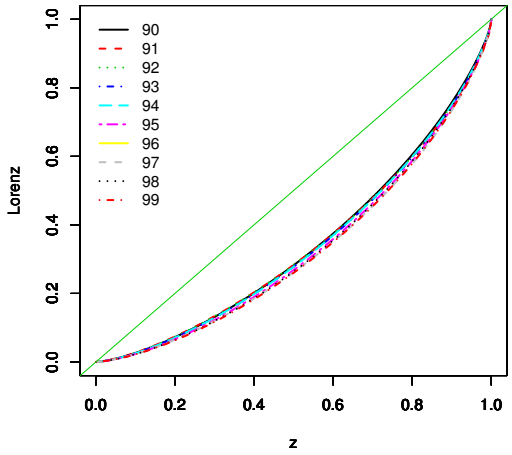


FIGURE 4. Lorenz curves for income data from 1390-1399.

These examples demonstrates the greater sensitivity of TTT curves in detecting subtle differences in income inequality, highlighting their practical advantage over Lorenz curves for distinguishing distributions with similar parameters.

4. Conclusion

This paper presented a quantile-based framework for the Total Time on Test transform, showing that classical reliability concepts such as hazard rates and aging measures can be reformulated in a clearer and unified form. Beyond revisiting known results, new inequalities and characterizations were established, enhancing the analytical power of TTT. The study further demonstrated that TTT is closely related to income inequality measures and can itself serve as a precise and reliable index of economic inequality. Numerical examples confirmed its greater sensitivity compared to conventional tools in distinguishing distributions. Overall, the findings reinforce TTT as a dual-purpose tool in both reliability analysis and inequality measurement.

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Mojtaba Esfahani
Department of Statistic,
Faculty of Basic Sciences and Engineering,
Velayat University,
Iranshahr, Iran
Email: m.esfahani@velayat.ac.ir

Mohammad Amini
Department of Statistics,
Faculty of Mathematical Sciences,
Ferdowsi University of Mashhad,
Mashhad, Iran
Email: m-amini@um.ac.ir

Gholam Reza Mohthashami Borzadar
Department of Statistics,
Faculty of Mathematical Sciences,
Ferdowsi University of Mashhad,
Mashhad, Iran
Email: grmohtashami@um.ac.ir



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