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IS GENERALIZATION IN MATHEMATICS TRULY ONE-DIRECTIONAL?

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This note is motivated by my recent publication in [1, Theorem B], whose surprising conclusion–that Euler's totient theorem and Fermat's little theorem can each be regarded as a generalization of the other–prompts the question in the tittle of the note. The theorem offers a sharpened and conceptually lucid refinement of Euler's classical theorem, relying on an improved exponent bound derived from the multiplicative order. Its structure naturally invites comparison with Carmichael's Theorem, which asserts that for any two natural numbers a and n, with $(a, n) = 1$, we have $a^{\lambda(n)} \equiv 1 \pmod{n}$, where $\lambda(n)$ is defined shortly. This result is traditionally regarded as Euler's totient theorem in its optimal exponent form. Yet the remarkable relationship between Theorem B and Carmichael's Theorem was not made explicit in the article. Before proceeding further, let us recall some definitions. Carmichael's function $\lambda(n)$ is defined as the least common multiple of $\lambda(p_i^{k_i})$, for $i = 1, 2, \ldots, r$, where $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of n. For a prime power: $\lambda(p_i^{k_i}) = p_i^{k_i - 1}(p_i - 1)$, where p_i is an odd prime, $\lambda(2^k) = 2^{k-2}$ for $k \geq 3$, and $\lambda(2) = 1$, $\lambda(4) = 2$. Let us also recall the well-known fact that if 2^k divides m, where k, m are positive integers, then $a^m \equiv 1 \pmod{2^{k+2}}$ if and only if a is odd, see also [1, Theorem A] for a more general result. The following is a restatement of Theorem B, in [1] with the added observation that one may also take $m = \lambda(n)$. For ease

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of reference, I shall call this version, **Connecting Theorem**, as it highlights the interplay between Euler's totient theorem, Carmichael's Theorem, and Fermat's little theorem.

Connecting Theorem ([1,Theorem B]). Let m, n be positive integers such that whenever p^k divides n, $p^{k-1}(p-1)$ divides m (resp., p^{k-2} divides m), where p is an odd prime number and k is a positive integer or $p = 2$ with $k \in \{1, 2\}$ (resp., where $p = 2$ and $k \geq 3$) (e.g., $m = \phi(n)$, where ϕ is the Euler's totient function of n, or $m = \lambda(n)$). Then $a^m \equiv 1 \pmod{n}$, where a is a positive integer with $(a, n) = 1$, if, and only if, Fermat's little theorem holds.

Proof. The proof of Theorem B in [1] may be applied word-for-word except for the case when 2^k divides n for $k \geq 3$, and hence 2^{k-2} divides m by our assumption, in which case, we may apply the above recalled fact in the introduction to see that $a^m \equiv 1 \pmod{2^k}$, which completes the proof in this case too. □

This connecting theorem, implies that both Euler's totient theorem and Carmichael's Theorem are direct consequences of Fermat's little theorem (note, in fact Carmichael's Theorem and Connecting Theorem are equivalent by definition). We notice that, $\lambda(n)$, is in general, a proper divisor of $\phi(n)$, and it is also the smallest possible integer for m in the above theorem, by definition of $\lambda(n)$. The minimality of $\lambda(n)$ in this role is inherent in its definition. Unlike the proof of Carmichael's theorem, which usually relies on Euler's totient theorem, the Chinese Remainder Theorem (CRT) and group theory, our connecting theorem similarly to [1, Theorem B] bypasses these tools. Their roles are effectively encoded within the "if and only if" structure of the theorem itself, see also [1], [2], and [3] for the power of the converse. Indeed, the theorem ensures that once the prime power conditions are met, the composite modulus automatically inherits the property. While the proofs of Theorem B in [1], and hence of our Connecting Theorem depend only on Fermat's little theorem, as their essential engine, the usual proofs of Carmichael's Theorem invoke Euler's totient theorem itself to extend the latter. This feature has led some authors to regard the extension from Euler's totient theorem to Carmichael's as conceptually less direct (similarly, to the extension of Pythagorean Theorem to Cosine formula (note, the earlier proofs of this formula were dependent on Pythagorean Theorem, see

[4])), despite the fact that Carmichael's Theorem is often described in the literature as a proper generalization of Euler's totient theorem. Thus, the connecting theorem manifestly shows, as it is also rightly emphasized in [1], that Euler's totient theorem and Fermat's little theorem are indeed genuine generalizations of one another (note, it also shows in fact all three classical theorems, Fermat's little Theorem, Euler's totient Theorem, and Carmichael's Theorem are naturally equivalent). The former bidirectional generalization is, to my knowledge, peculiar and unprecedented (note, it shows unmistakably that Fermat's little Theorem is, indeed, a genuine generalization of Euler's totient theorem, overturning the belief–held since Euler's time–that only the converse is true). I have not found another pair of distinct mathematical theorems in the literature where the notion of generalization applies in both directions. Since the publication of [1], I have been struck by the feeling that this phenomenon cannot be unique in mathematics. Yet my searches across number theory, algebra, geometry, and logic– including categorical correspondence and equivalent reformulations–have not revealed another example so far. If readers are aware of similar examples, I would be very interested to learn of such cases.

As I have sought to make clear above, my intention is not to dispute the values of these classical results, but rather to revisit them in light of certain observations that may enrich their understanding. Finally, by drawing attention to these points, I hope this commentary, not only contributes constructively to the ongoing dialogue around these classical results and their place within the broader mathematical context, but also inspires others to look for analogous patterns elsewhere.

References

[1] O.A.S. Karamzadeh, *Euler's Totient theorem and Fermat's little theorem are generalization of one another*, Mat. Gaz. 572 (July 2024), 314-316.

[2] O.A.S. Karamzadeh, *If, and only if, in mathemaics*, Eur. Math. Soc. Mag. (2024), published online first, DOI 10.4171/MAG/183.

[3] O.A.S. Karamzadeh. *On the converse of Johnson's three circles theorem*, Mat. Gaz. To appear 2926.

[4] Artin. Sharifi, *An Effective Tool Might Lead us on to an Ineffective proof*, The Mathematical Intelligencer, 33, (2016), 1-2.

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