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MODULES WITH DUAL ENOUGH NOETHERIANS

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Abstract. In this paper, we investigate a class of modules in which every factor module has a nonzero Artinian factor module, referring to them as *modules with dual enough Noetherians*. Additionally, we introduce and study *dual seminoetherian modules*, which are modules where every nonzero submodule has a nonzero Artinian factor module. Our work serves as a dual counterpart to the results of Kourki and Tribak in Communications in Algebra (2022). We establish several properties of these classes of modules. Notably, we prove that the class of modules with dual enough Noetherians is closed under factor modules, while the class of dual seminoetherian modules is closed under submodules. These findings contribute to the broader structural theory of module categories and extend existing duality principles in module theory.

1. Introduction and Preliminaries

Throughout this article, all rings are associative with identity and all modules are unital right modules. Let M be an R-module and let $N \subseteq M$. The notation $N \leq M$ indicates that N is a submodule of M. Lemonnier [2] has introduced the concepts of deviation and codeviation of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R give the concepts of Krull dimension (in the sense of Rentschler and Gabriel) and the dual Krull dimension of M, respectively. Recall that,

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the Krull dimension (resp., Noetherian dimension) measures how close a module is to being Artinian (resp., Noetherian). For the reader's convenience, we now recall these definitions.

The *Krull dimension* of a module was defined by Gordon and Robson [11] in the following way.

Definition 1.1. Let M be a right R-module. The Krull dimension of M, which will be denoted by $k\text{-dim}\, M$, is defined by transfinite recursion as follows:

(1) if $M = 0$, then $k\text{-dim}\, M = -1$;
(2) if α is an ordinal number and $k\text{-dim}\, M \not< \alpha$, then $k\text{-dim}\, M = \alpha$, provided there is no infinite descending chain $M = M_0 \geq M_1 \geq ...$ of submodules M_i such that $k\text{-dim}\,(M_{i-1}/M_i) \not< \alpha$ for all $i \in N$;
(3) it is possible that there is no ordinal α such that $k\text{-dim}\, M = \alpha$. In this case, M has no Krull dimension.

Following [9], the *Noetherian dimension* (the terms *dual Krull dimension* [12], and *N-dimension* [8] are also used) of a module is defined inductively in the following way.

Definition 1.2. Let M be a right R-module. The Noetherian dimension of M, which will be denoted by $n\text{-dim}\ M$, is defined as follows:

(1) if $M = 0$, then $n\text{-dim}\ M = -1$;
(2) if α is an ordinal number and $n\text{-dim}\ M \not< \alpha$, then $n\text{-dim}\ M = \alpha$, provided there is no infinite ascending chain $M_0 \leq M_1 \leq ...$ of submodules M_i such that $n\text{-dim}\ (M_i/M_{i-1}) \not< \alpha$ for all $i \in N$;
(3) it is possible that there is no ordinal α such that $n\text{-dim}\ M = \alpha$. In this case, M has no Noetherian dimension.

In [5], Kourki and Tribak introduced the following concepts.

Definition 1.3. ([5, Definitions 2.1 and 3.1]) Let M be an R-module.

(1) We say that M *has enough Noetherians* if any nonzero submodule of M contains a nonzero Noetherian submodule.
(2) We say that M is *seminoetherian* if any nonzero factor module of M contains a nonzero Noetherian submodule.

In this paper we define the dual notions as follow as follows:

Definition 1.4. Let M be an R-module.

(1) We say that M *has dual enough Noetherians* if any nonzero factor module of M contains a nonzero Artinian factor module.
(2) We say that M is *dual seminoetherian* if any nonzero submodule of M contains a nonzero Artinian factor module.

Let us give a brief outline of this paper. Section 1 is the introduction. In Section 2, we introduce the notion of modules having dual enough Noetherians. We show that any Artinian module has dual enough noethrians but the converse is not true in general, see Example 2.2. We recall that, a nonzero module M is called co-isosimple if it is isomorphic to all its nonzero quotients, see [1]. We show that, if M is a co-isosimple module, then M is Artinian if and only if M has dual enough Noetherians, see Proposition 2.4. We recall that the Goldie dimension of an R-module M, denoted by G-dim M is the supremum λ of all cardinals k such that M contains the direct sum of k nonzero submodules. We also recall that by a quotient finite dimensional module M we mean that for each submodule N of M, $\frac{M}{N}$ has finite Goldie dimension. We observe that, if every nonzero submodule of a quotient finite dimensional module M has dual enough Noetherians, then M has Noetherian dimension, see Proposition 2.12.

Section 3 is devoted to the study of dual seminoetherian modules, which is the dual of the concept of seminoetherian modules, see [5]. We investigate some basic properties of dual seminoetherian modules. We show that, a quotient finite dimensional module M is Artinian if and only if every nonzero factor module of M is dual seminoetherian, see Proposition 3.11. We also show that, any dual seminoetherian R-module is max-module if and only if R is AN-ring, see Proposition 3.12.

The reader is referred to [5, 11, 12], for the necessary background not explicitly given here.

2. DUAL ENOUGH NOETHERIANS

In this section, we introduce and study the concept of dual enough Noetherians. Next, we give our definition of dual enough Noetherians.

Definition 2.1. We say that an R-module M has dual enough Noetherians, if any nonzero factor module of M has a nonzero Artinian factor module. A ring R is said to have dual enough Noetherians, if R as an R-module, has dual enough Noetherians.

Any Artinian module has dual enough Noetherians. But the converse is not true in general. See the following examples.

Example 2.2. Let $M = \bigoplus_{i\in I} S_i$, where each S_i is a simple module and I has infinite cardinal number. Then M is not Artinian, but it has dual enough Noetherians, see Corollary 2.8.

Example 2.3. The $\mathbb{Z}$-module $\mathbb{Z}$ has dual enough Noetherians, although it is not Artinian.

Recall that an R-module M is called co-isosimple if it is isomorphic to all its nonzero quotients, see [1].

Proposition 2.4. *Let M be a co-isosimple module. Then M is Artinian if and only if M has dual enough Noetherians.*

Proof. ($\Rightarrow$) If M is Artinian, then by definition M has dual enough Noetherians, see Example 2.2.

($\Leftarrow$) Suppose that M has dual enough Noetherians. Let $N \leq M$, then there exists $N \leq K < M$ such that $\frac{M}{K}$ is Artinian module. Since M is co-isosimple, we have $M \cong \frac{M}{K}$. Hence M is Artinian. □

Proposition 2.5. *Let M be an R-module. If M has dual enough Noetherians, then so does any factor module of M.*

Proof. Let $\frac{M}{N}$ be any factor module of M. If $\frac{K}{N}$ is a proper submodule of $\frac{M}{N}$, then $N \leq K < M$. Since $K < M$ and M has dual enough Noetherians, so there exists $K \leq T < M$ such that $\frac{M}{T}$ is Artinian. Hence $\frac{K}{N} \leq \frac{T}{N} < \frac{M}{N}$ and $\frac{\frac{M}{N}}{\frac{T}{N}} \cong \frac{M}{T}$ is Artinian, and we are done. □

Recall that an R-module M is called coatomic if each proper submodule is contained in a maximal submodule of M, see [1]. We also recall that a ring R is called amen, if Artinian modules equals Noetherian, see [7].

Lemma 2.6. *Every coatomic module has dual enough Noetherians. If R is an amen ring, the converse is true.*

Proof. Let M be a coatomic module and $N < M$. Since M is coatomic, we infer that there is a maximal submodule $N \leq T < M$. Therefore $\frac{M}{T}$ is simple and so it is Artinian. Now let R be an amen ring. Let $N < M$. Since M has dual enough Noetherians, so there exists $N \leq K < M$ such that $\frac{M}{K}$ is Artinian. Since R is an amen ring, we infer that $\frac{M}{K}$ is Noetherian. Hence there is a maximal submodule $\frac{T}{K} < \frac{M}{K}$. So T is a maximal submodule of M which contains N. Therefore M is a coatomic module. □

In view of Lemma 2.6, we have the following corollaries.

Corollary 2.7. *Every finitely generated R-module M, has dual enough Noetherians.*

Corollary 2.8. *Every semisimple R-module M, has dual enough Noetherians.*

Corollary 2.9. *Any commutative ring R, has dual enough Noetherians.*

In the next result, we investigate the relationship between the Noetherian dimension of a quotient finite dimensional module M and having enough Noetherians. First we recall the following result.

Proposition 2.10. *An R-module has Noetherian dimension if and only if it has Krull dimension.*

We now recall the following well known and important result due to Lemmonier, see [3, Theorem 2.4] and [12, Proposition 2.2].

Proposition 2.11. *The following statements are equivalent for any R-module M and any ordinal $\alpha \geq 0$:*

(1) $n\text{-dim}\, M \leq \alpha$
(2) *M is quotient finite dimensional and for any $N < P \leq M$, there exists X with $N \leq X < P$ with $n\text{-dim}\, \frac{P}{X} \leq \alpha$.*

Now, in view of Proposition 2.11 and Proposition 2.5, the following proposition is evident.

Proposition 2.12. *Let M be a quotient finite dimensional module. If every nonzero submodule of M has dual enough Noetherians, then M has Noetherian dimension.*

Proof. Let $N < P \leq M$. In view of our hypothesis and Proposition 2.5, we get $\frac{P}{N}$ has dual enough Noetherians, thus there exists a nonzero Artinian factor module $\frac{P}{X}$ of $\frac{P}{N}$. In view of Proposition 2.10, we infer that $\frac{P}{X}$ has Noetherian dimension. By Proposition 2.11 we get the result. □

Let M be an R-module. We recall that $R(+)M = \{(r, m) \mid r \in R, m \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity $(1, 0)$ and it is called the idealization of M. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. When $I(+)N$ is an ideal, $\frac{M}{N}$ is an $\frac{R}{I}$-module and $\frac{(R(+)M)}{(I(+)N)} \cong (\frac{R}{I})(+)(\frac{M}{N})$. In particular, $\frac{R(+)M}{0(+)N} \cong R(+)(\frac{M}{N})$ and therefore $\frac{R(+)M}{0(+)M} \cong R$, see [4].

Theorem 2.13. *([4, Theorem 4.8]) Let R be a commutative ring and M be an R-module. Then $R(+)M$ Noetherian(resp., Artinian) if and only if R is Noetherian(resp., Artinian) and M is finitely generated.*

Proposition 2.14. *Let M be a finitely generated R-module and $T = R(+)M$. Then T has dual enough Noetherians if and only if R has dual enough Noetherians.*

Proof. ($\Leftarrow$) Let $I(+)N$ is a proper ideal of T. If I is a proper ideal of R, then since R has dual enough Noetherians, so there exists $I \leq J < R$ such that $\frac{R}{J}$ is Artinian. Therefore $I(+)N \leq J(+)M < R(+)M$ and $\frac{R(+)M}{J(+)M} \cong \frac{R}{J}$ is Artinian. Otherwise N is a proper submodule of M. Since M is finitely generated, we infer that $\frac{M}{N}$ is a nonzero finitely generated R-module. Hence $\frac{R(+)M}{R(+)N} \cong \frac{M}{N}$ and we are done, see Theorem 2.13.
($\Rightarrow$) Let I be a proper ideal of R. Then $I(+)M$ is a proper ideal of T. Since T has dual enough Noetherians, we infer that, there exists $I(+)M \leq J(+)M < R(+)M$ such that $\frac{R(+)M}{J(+)M}$ is Artinian. But $\frac{R(+)M}{J(+)M} \cong \frac{R}{J}$ is Artinian. □

3. DUAL SEMINOETHERIAN

In this section, we introduce the concept of dual seminoetherian modules, which is the dual of seminoetherian modules as introduced in [5].

Definition 3.1. An R-module M is said to be dual seminoetherian if every nonzero submodule of M possesses a nonzero Artinian factor module. A ring R is called dual seminoetherian if R, as an R-module, has this property.

We cite the following definition from [10].

Definition 3.2. A nonzero R-module M is called a max-module (or seminoetherian) if any nonzero submodule of M contains a maximal submodule.

Remark 3.3. Any max-module is a dual seminoetherian module. But the converse is not true in general. For example, $\mathbb{Z}$-module $\mathbb{Z}_{P^\infty}$ is Artinian, and therefore it is dual seminoetherian. But it is not a max-module, see [10, Proposition 3.2].

If R is an amen ring, then the converse of Remark 3.3 is true. Let $N \leq M$. Since M is dual seminoetherian, so there exists $K < N \leq M$ such that $\frac{N}{K}$ is Artinian and so it is Noetherian. Hence $\frac{N}{K}$ has a maximal submodule, $\frac{T}{K}$ say. This implies that T is a maximal submodule of N.

Remark 3.4. Any Noetherian module is a max-module. Hence any Noetherian module is dual seminoetherian.

The converse of Remark 3.4 is not true in general. For example, $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is dual seminoetherian, but it is not Noetherian.

Remark 3.5. Any Artinian module M, is dual seminoetherian. But the converse is not true in general, for example $\mathbb{Z}$-module $\mathbb{Z}$ is Noetherian so it is dual seminoetherian, but it is not Artinian.

Proposition 3.6. *Let M be a dual seminoetherian module. If N is a nonzero submodule of M, then N also is dual seminoetherian.*

Proof. Let K be a nonzero submodule of N, then K is a nonzero submodule of M. Since M is dual seminoetherian, so there exists a proper submodule T of K such that $\frac{K}{T}$ is Artinian and we are done. □

Proposition 3.7. *Let M be a finitely generated R-module. Then M is dual seminoetherian if and only if N is dual seminoetherian for each proper submodule N of M.*

Proof. ($\Rightarrow$) It is obvious, see 3.6.
($\Leftarrow$) Let $0 \neq N \leq M$. If $N = M$, since M is finitely generated, so there exists a maximal submodule $K < M$ such that $\frac{M}{K}$ is simple and hence it is Artinian. Otherwise let $N < M$. Since N is dual seminoetherian so there exists $K < N$ such that $\frac{N}{K}$ is Artinian. Therefore M is dual seminoetherian. □

In the following results, we investigate the relationship between the concepts of having dual enough noetherias and being dual seminoetherian.

Proposition 3.8. *Let M be an R module. If every proper submodule of M has dual enough Noetherians, then M is dual seminoetherian.*

Proof. Let P be a nonzero submodule of M. Then, since P has dual enough Noetherians, there exists a nonzero Artinian factor module $\frac{P}{N}$ of P. Hence we are done. □

Proposition 3.9. *Let M be an R module. If every nonzero factor module of M is dual seminoetherian, then M has dual enough Noetherians.*

Proof. Let P be a nonzero submodule of M. Then, since $\frac{M}{P}$ is dual seminoetherian, there exists a nonzero Artinian submodule $\frac{P}{N}$ of $\frac{M}{N}$. Hence we are done. □

We recall the following result from [11].

Proposition 3.10. *The following are equivalent for a module M and an ordinal number α:*

(1) $k\text{-dim}\, M \leq \alpha$.
(2) *M is quotient finite dimensional module and for any $N < M$ there exists P such that $N < P \leq M$ and $k\text{-dim}\, \frac{P}{N} \leq \alpha$.*

Proposition 3.11. *A quotient finite dimensional module M is Artinian if and only if every nonzero factor module of M is dual seminoetherian.*

Proof. Let $N < M$. In view of our hypothesis, we get $\frac{M}{N}$ is dual seminoetherian, thus there exists a nonzero Artinian submodule $\frac{P}{N}$ of $\frac{M}{N}$. Hence $\frac{P}{N}$ has Krull dimension and $k\text{-dim}\, \frac{P}{N} \leq 0$. In view of Proposition 3.10, we infer that M is Artinian. □

We recall that a ring R is called AN, if any Artinian module is Noetherian.

Proposition 3.12. *The following are equivalent for a ring R:*

(1) *Any dual seminoetherian R-module, is a max-module;*
(2) *R is AN-ring.*

Proof. (1) $\Rightarrow$ (2) Let M be an Artinian R-module. Then M is dual seminoetherian, so by assumption M is a max-module. Hence M is Noetherian, see [6, Proposition 1.1].

(2) $\Rightarrow$ (1) Let M be a dual seminoetherian module, and let $N \leq M$. So there exists $K < N \leq M$ such that $\frac{N}{K}$ is Artinian. Then $\frac{N}{K}$ is Noetherian. Hence $\frac{N}{K}$ has a maximal submodule, $\frac{T}{K}$ say. Therefore T is a maximal submodule of N. □

Theorem 3.13. *Let R be a ring and let M be a Noetherian R-module. If R is a dual seminoetherian ring, then so does $T = R(+)M$.*

Proof. Let $I(+)N$ be a nonzero ideal of T. So I is an ideal of R and N is a submodule of M. If $I \neq 0$, since R is dual seminoetherian, so there exists $J < I \leq R$ such that $\frac{I}{J}$ is Artinian. Therefore $J(+)N < I(+)N \leq R(+)M$ and $\frac{I(+)N}{J(+)N} \cong \frac{I}{J}$ is Artinian. Otherwise, since M is Noetherian, we infer that N is finitely generated R-module. Hence $0(+)N$ is Artinian and we are done, see Theorem 2.13. □

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