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ANALYTICAL AND NUMERICAL STUDY OF A CLASS OF NONLINEAR PARABOLIC MOVING BOUNDARY PROBLEMS

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Abstract. The main contribution of this article is the investigation of a parabolic problem with a moving boundary and nonlinear boundary conditions. Because of the non-linearity of the problem, it is important to put a lot of effort into providing efficient numerical schemes. For this purpose, a numerical approach based on the finite difference method with variable time steps is developed. First, based on the front-fixing approach, the problem is derived as a nonlinear fixed domain parabolic problem, and the uniqueness of the solution of the problem is proved. In sequence, a variable time step finite difference method is established to find the solution of the fixed domain problem. Finally, some numerical results are presented for two test problems to prove the ability and strength of the numerical method.

1. Introduction

Mathematical modeling and simulation have received great attention in various fields of science, including engineering, physics, biology, and industrial applications [1, 2, 3, 4, 5, 6, 7, 8, 9]. In various fields, phenomena are modeled by moving boundary problems of partial differential equations, especially parabolic equations [10, 11, 12, 13, 14, 15]. These problems typically arise in processes such as heat conduction [16], phase transitions[17],

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material solidification [18], drug release [19], and cancer growth modeling [20]. In many real-world applications, moving boundary problems in the modeling of thermal diffusion, fluid mechanics, and porous media flows, where the boundary of the problem changes dynamically over time [21, 22].

The concept of moving boundary problems dates back to the classical Stefan problem [23], which describes phase changes, such as the melting of ice. The Stefan problem is widely recognized as a foundational example of a moving boundary problem. Since its introduction, numerous mathematical techniques have been developed to address similar problems in different fields. Over the years, several approaches have been developed to solve these problems numerically and analytically. Notable contributions in this area include the front-fixing method, moving mesh techniques, finite difference and finite element methods, and phase field approaches [14, 24, 25, 26, 27, 28, 29]. Crank [30] and Tarzia [31] have provided extensive studies on moving boundary problems governed by the heat equation. More recently, studies have explored the application of these problems in drug release mechanisms and material diffusion [19, 20, 32, 33, 34]. Due to important physical and chemical properties such as diffusion, solubility, erosion, flow, etc., moving boundary problems arise when a device encounters a diffusing solvent [35, 15].

Many industrial and biomedical applications involve nonlinear moving boundary problems where the diffusion properties and external forces alter the evolution of the boundary. For example, in drug delivery systems, diffusion-based transport mechanisms and swelling behavior influence the release rate of active pharmaceutical ingredients [10, 36]. Similarly, in material sciences, solidification and phase transformation processes require accurate mathematical models to predict the interface dynamics between different phases [37, 38, 39]. Addressing these challenges requires robust numerical methods that can accurately capture the temporal evolution of the moving boundary while ensuring computational efficiency [40].

The main objective of this study is to develop and analyze a class of nonlinear parabolic moving boundary problems. The main feature of the moving boundary value problems is that the domain in which the PDE is defined changes over time, which adds complexity to the solution process. Therefore, by transforming the original problem into a fixed-domain formulation using a front-fixing approach, we then propose a numerical method based on the finite difference approach to approximate the solution efficiently. The novelty of this

work lies in identifying the uniqueness of the solution and a time variable approach to find the unknown moving boundary. The results demonstrate the proposed approach's ability to deal with nonlinear moving boundary problems effectively.

The structure of this paper is as follows

Section 2 presents our interest parabolic moving boundary problem and describes the transformation into a fixed domain problem. Section 3 discusses the uniqueness of the solution. Section 4 presents the proposed numerical method, including the numerical and analytical solution approach and implementation details. Finally, Section 5 provides numerical results for selected test problems, illustrating the robustness and accuracy of the proposed method.

2. Problem Definition

Based on mathematical models of many phenomena, in this section, we focus on an important class of nonlinear moving boundary problems. Consider the following moving boundary parabolic problem of determining $u(x,t)$ and the moving boundary function $s(t)$ as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) + \rho(u) + f(x,t), \quad 0 < x < s(t) \leq 1, \quad 0 < t < T_f, \tag{2.1}$$

$$\alpha_{11}\left(D\frac{\partial u}{\partial x}\right)\Bigg|_{x=0} - \alpha_{12}u(0,t) = g(t), \tag{2.2}$$

$$\alpha_{21}\left(D\frac{\partial u}{\partial x}\right)\Bigg|_{x=s(t)} + \alpha_{22}u(s(t),t) = h(t), \tag{2.3}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq s_0, \tag{2.4}$$

$$\frac{ds}{dt} = F(s(t), u(s(t),t)) \quad 0 < t < T_f, \tag{2.5}$$

$$s(0) = s_0. \tag{2.6}$$

The following assumptions are considered associated with the problem.

I. The diffusion coefficient D is considered a positive differentiable function of x and t, $D = D(x,t)$.

II. $g,\ h,\ u_0$, and f are L_2 functions, F is a differentiable function and $\alpha_{ij}; i,j = 1,2$ are known parameters.

III. ρ is a Lipschitz continuous function with Lipschitz constant β, i.e., $\|\rho(u_1)-\rho(u_2)\| \leq \beta\|u_1 - u_2\|$.

To analyze the problem (2.1)-(2.6), based on a front-fixing approach, first we should consider a new variable $\tilde{X} = x/(s(t))$. Suppose

$$\tilde{X} = \frac{x}{s(t)}, \tilde{u} = u(\tilde{X}s(t),t), \tilde{D} = D(\tilde{X}s(t),t), \tilde{\rho} = \rho(\tilde{u}), \tilde{f} = f(\tilde{X}s(t),t), \tilde{F} = F(s,\tilde{u}). \tag{2.7}$$

Considering (2.7), one may find the new version of problem (2.1)-(2.6). For simplicity, let's drop the tilde sign of variables and use x instead of X, and write the fixed-domain problem as follows

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)\frac{1}{s^2(t)} + \frac{xs'(t)}{s}u_x + \rho(u) + f, \quad x \in (0,1), \quad t \in (0,T_f), \tag{2.8}$$

$$\frac{1}{s(t)}\alpha_{11}\left(D\frac{\partial u}{\partial x}\right)\Bigg|_{x=0} - \alpha_{12}u(0,t) = g(t), \tag{2.9}$$

$$\frac{1}{s(t)}\alpha_{21}\left(D\frac{\partial u}{\partial x}\right)\Bigg|_{x=1} + \alpha_{22}u(1,t) = h(t), \tag{2.10}$$

$$u(x,0) = u_0(x), \tag{2.11}$$

$$\frac{ds}{dt} = F(s,u(s,t)), \tag{2.12}$$

$$s(0) = s_0 \qquad t_0 = 0. \tag{2.13}$$

3. Uniqueness of solution

Our aim in this section is to prove that the solution of problem (2.8)-(2.13) is unique. We would like to clarify that the problem addressed in this paper originates from a well-posed physical model governed by conservation laws. In such contexts, the existence of solutions is often expected based on physical intuition and prior modeling frameworks. One may find that some foundational studies in the literature have addressed existence results for nonlinear moving-boundary problems of mixed type, offering critical insights that underpin the theoretical framework of our work, such as [41]. It is very helpful to write the problem (2.8)-(2.13) in a week form. To this end, one may use a test function such as $\varphi(x,t) \in H_1[0,1] \times C^1(0,T_f)$. Applying the inner product $<,>$ of the space $L_2[0,1]$ with respect to spatial variable x yields

$$\langle u_t, \varphi\rangle = \frac{1}{s^2(t)}\langle \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right), \varphi\rangle + \frac{s'(t)}{s(t)}\langle x\frac{\partial u}{\partial x}, \varphi\rangle + \langle \rho(u), \varphi\rangle + \langle f_1, \varphi\rangle, \tag{3.1}$$

The expression $\langle \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right), \varphi\rangle$ can be expanded as

$$\langle \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right), \varphi\rangle = \int_0^1 \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)\varphi\, dx = D\frac{\partial u}{\partial x}\varphi\Big|_0^1 - \int_0^1 D\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\, dx. \tag{3.2}$$

Using boundary conditions (2.9) and (2.10), one may find

$$\left\langle \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right), \varphi\right\rangle = -\frac{s(t)}{\alpha_{11}}g(t) - \frac{\alpha_{12}}{\alpha_{11}}s(t)u(0,t) + \frac{h(t)s(t)}{\alpha_{21}} - \frac{\alpha_{22}}{\alpha_{21}}s(t)u(1,t) - \langle Du_x, \varphi_x\rangle. \tag{3.3}$$

Then we have

$$\begin{aligned}\langle u_t, \varphi\rangle = &-\frac{1}{s(t)^2}\langle Du_x, \varphi_x\rangle + \frac{s'(t)}{s(t)}\langle xu_x, \varphi\rangle + \langle \rho(u), \varphi\rangle + \langle f, \varphi\rangle \\ &+ s(t)\left[\frac{h(t)}{\alpha_{21}} - \frac{\alpha_{22}}{\alpha_{21}}u(1,t)\right] - s(t)\left[\frac{g(t)}{\alpha_{11}} + \frac{\alpha_{12}}{\alpha_{11}}u(0,t)\right],\end{aligned} \tag{3.4}$$

$$u(x,0) = u_0(x), \tag{3.5}$$

$$\frac{ds}{dt} = F(s, u(s,t)). \tag{3.6}$$

Now suppose that (u_1, s_1) and (u_2, s_2) are two solutions for problem (3.4)-(3.6). Write two equations (3.3)-(3.5) for u_1 and u_2, and subtracting them from together and using $W = u_1 - u_2$ and $S_w = s_1 - s_2$, yields

$$\begin{aligned}\langle w_t, \varphi\rangle &= \left\langle -\frac{1}{s_1^2} D_1 w_x, \varphi_x \right\rangle + \left\langle \left(-\frac{D_1}{s_1^2} + \frac{D_2}{s_2^2} \right) u_2 x, \varphi_x \right\rangle \\ &+ \left\langle \frac{s_1'}{s_1} x w_x, \varphi \right\rangle + \left\langle \left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2} \right) x u_2 x, \varphi \right\rangle \\ &+ \langle \rho(u_2) - \rho(u_1), \varphi\rangle + \langle f_1 - f_2, \varphi\rangle \\ &+ \frac{s_w g(t)}{s_1 s_2} \frac{1}{\alpha_{11}} + \frac{\alpha_{12}}{\alpha_{11}} \frac{s_w u_2(0,t) - s_2 w(0,t)}{s_1 s_2} \\ &- \frac{s_w h(t)}{s_1 s_2} \frac{1}{\alpha_{21}} - \frac{\alpha_{22}}{\alpha_{21}} \frac{s_w u_2(1,t) - s_2 w(1,t)}{s_1 s_2},\end{aligned} \tag{3.7}$$

where $D_i = D(x s_i(t), t)$ and $f_i = f(x s_i(t), t)$; i=1,2. Now suppose $\varphi = u_1 - u_2 = w$, then we have

$$\begin{aligned}\langle w_t, w\rangle &= \left\langle -\frac{1}{s_1^2} D_1 w_x, w_x \right\rangle + \left\langle \left(\frac{D_2}{s_2^2} - \frac{D_1}{s_1^2} \right) u_2 x, w_x \right\rangle \\ &+ \left\langle \frac{s_1'}{s_1} x w_x, w \right\rangle + \left\langle \left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2} \right) x u_2 x, w \right\rangle \\ &+ \langle \rho(u_2) - \rho(u_1), w\rangle + \langle f_1 - f_2, w\rangle \\ &+ \frac{s_w}{s_1 s_2} \frac{g(t)}{\alpha_{11}} + \frac{\alpha_{12}}{\alpha_{11}} \frac{s_w u_2(0,t) - s_2 w(0,t)}{s_1 s_2} \\ &- \frac{h(t)}{\alpha_{21}} \frac{s_w}{s_1 s_2} + \frac{\alpha_{22}}{\alpha_{21}} \frac{-s_w u_2(1,t) + s_2 w(1,t)}{s_1 s_2}.\end{aligned} \tag{3.8}$$

Using Mean Value Theorem (MVT), we can conclude

$$\left\langle -\frac{1}{s_1^2} D_1 w_x, w_x \right\rangle = -\frac{D_1(\xi, t)}{s_1^2} \|w_x\|^2, \tag{3.9}$$

Furthermore, if we suppose

$$\eta(s) = \frac{D(s,t)}{s^2}, \tag{3.10}$$

then we may write

$$\eta(s_1) - \eta(s_2) = \eta'(\xi)(s_1 - s_2), \tag{3.11}$$

where $\eta'(\xi)$ is a continuous function.

Now, using the following inequality (Young's inequality)

$$\langle k_1, k_2\rangle \leq \frac{\epsilon}{2}\|k_1\|^2 + \frac{2}{\epsilon}\|k_2\|^2, \tag{3.12}$$

one may conclude that

$$\begin{aligned}
\langle w, w_t\rangle = &-\frac{D_2(\xi,t)}{s_1^2}\|w_x\|^2 + \frac{2}{\epsilon_1}\|\eta'(\xi)\|^2\|u_2x\|^2 s_w^2 + \frac{\epsilon_1}{2}\|w_x\|^2 \\
&+ \frac{2}{\epsilon_2}\left(\frac{s_1'}{s_1}\right)^2\|w\|^2 + \frac{\epsilon_2}{2}\|w_x\|^2 + \frac{\epsilon_3}{2}\|u_2x\|^2 \\
&+ \frac{2}{\epsilon_3}\left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2}\right)^2\|w\|^2 + \frac{\epsilon_4}{2}\|\rho(u_1) - \rho(u_2)\|^2 + \frac{2}{\epsilon_4}\|w\|^2 \\
&+ \frac{\epsilon_5}{2}\|f_1 - f_2\|^2\|w\|^2 + \frac{2}{\epsilon_6}\left(\frac{g(t)}{s_1s_2}\right)s_w^2 + \frac{\epsilon_6}{2} \\
&+ \frac{2}{\epsilon_7}\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^2\left(\frac{u_2(0,t)}{s_1s_2}\right)^2 s_w^2 + \frac{\epsilon_7}{2} + \frac{\epsilon_8}{2} \\
&+ \frac{2}{\epsilon_8}\left(\frac{\alpha_{12}}{\alpha_{11}s_1}\right)^2\|w\|^2 + \frac{\epsilon_8}{2} + \frac{2}{\epsilon_9}\left(\frac{h(t)}{s_1s_2}\right)s_w^2 + \frac{2}{\epsilon_{10}}\left(\frac{\alpha_{22}}{\alpha_{21}}\right)^2 \\
&\times\left(\frac{u_2(1,t)}{s_1s_2}\right)^2 s_w^2 + \frac{\epsilon_{10}}{2} + \frac{\epsilon_{11}}{2} + \frac{2}{\epsilon_{11}}\left(\frac{\alpha_{22}}{\alpha_{21}s_1}\right)^2\|w\|^2.
\end{aligned} \tag{3.13}$$

On the other hand, we suppose $F = F(s,u)$ is a differentiable function of s and u, then one can find that $F(s(t), u(s(t),t))$ is a differentiable function of s, such as $\phi(s)$ and using the MVT yields. therefore

$$\frac{ds_1}{ds_2} - \frac{ds_2}{ds_2} = F(s_1, u_1(s_1,t)) - F(s_2, u_2(s_2,t)) = \phi(s_1) - \phi(s_2) = \phi'(\xi)(s_1 - s_2). \tag{3.14}$$

Then we see

$$\frac{d}{dt}s_w = \phi'(\xi)\cdot s_w. \tag{3.15}$$

Multiplying equation (3.15) by s_w, we can write

$$s_w \cdot \frac{d}{dt} s_w = \frac{d}{dt} \frac{s_w^2}{2} = \phi'(\xi) \cdot s_w^2 \leq \mu_1 s_w^2, \tag{3.16}$$

Finally, by using equations (3.12) and (3.15), we find

$$\begin{aligned} \frac{d}{dt}\left(s_2 + \|w\|^2\right) \leq & \left(-\frac{D(\xi,t)}{s_1^2} + \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2}\right)\|w\|^2 + \frac{2}{\epsilon_2}\left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2}\right)^2 \\ & + \frac{\epsilon_4}{2}\mu_2 + \frac{2}{\epsilon_4} + \frac{2}{\epsilon_5}\|f_1 - f_2\|^2 + \frac{2}{\epsilon_8}\left(\frac{\alpha_{12}}{\alpha_{11}s_1}\right)^2 \\ & + \frac{2}{\epsilon_{11}}\left(\frac{\alpha_{22}}{\alpha_{21}s_1}\right)^2\|w\|^2 + \frac{2}{\epsilon_1}\|\eta'(\xi)\|^2\|u_2 x\|^2 \\ & + \frac{2}{\epsilon_6}\left(\frac{g(t)}{s_1 s_2}\right)^2 + \frac{2}{\epsilon_7}\left(\frac{\alpha_{12}}{\alpha_{11}}\right)^2\left(\frac{u_2(0,t)}{s_1 s_2}\right)^2 \\ & + \frac{2}{\epsilon_9}\left(\frac{h(t)}{s_1 s_2}\right)^2 + \frac{2}{\epsilon_{10}}\left(\frac{\alpha_{22}}{\alpha_{21}}\right)^2\left(\frac{u_2(1,t)}{s_1 s_2}\right)^2 \\ & + \mu_1 s_w^2 + \frac{\epsilon_3}{2}\|u_2 x\|^2 + \frac{\epsilon_6}{2} + \frac{\epsilon_7}{2} + \frac{\epsilon_8}{2} + \frac{\epsilon_9}{2} + \frac{\epsilon_{10}}{2} + \frac{\epsilon_{11}}{2}. \end{aligned} \tag{3.17}$$

The constants $\epsilon_i; i = 1, 2, \ldots, 11$ are arbitrary, so one may consider ϵ_1 and ϵ_2 such that the coefficient of $\|w\|^2$ becomes negative. Therefore, the first term on the right-hand side of equation (3.17) can be omitted. In addition, the last seven terms are coefficients of ϵ_i and can also be omitted. Finally, it is clear that the coefficients of $\|w\|^2$ and s_w^2 are L_2 functions and can be bounded by a function such as $\alpha(t)$. Hence, we can conclude

$$\frac{d}{dt}\left(\|w\|^2 + s_w^2\right) \leq \alpha(t)\left(\|w\|^2 + s_w^2\right). \tag{3.18}$$

On the other hand, regarding the initial condition, $w(x,0) = s_w(0) = 0$, and by applying Gronwall's inequality, we conclude that $\|w\|^2 = s_w^2 \equiv 0$.

This shows the uniqueness of the solution. Now we can summarize the above analysis in the following statement.

Theorem 1. For the Problem (2.1)-(2.6), consider that the functions f, g, h and μ are L_2 functions, $\alpha > 0$ and F is differentiable. Then the problem (2.8)-(2.13) has a unique solution for $u \in H_1[0,1] \times C^1(0, T_f)$ and $s \in C^1(0, T_f)$.

4. Numerical Approach

This section establishes a numerical approach for solving the fixed-domain problem (2.8)-(2.13). To this end, a variable time step method is proposed to numerically solve the problem as follows.

First, we divided the spatial interval $[0, 1]$ into M equidistant sub-intervals with an arbitrary step length as $\Delta x = \frac{1}{M}$. Now suppose t_1 is the time that $s(t_1)$ meets x_1,i.e; $s(t_1) = x_1$, t_2 is the time that $s(t_2) = x_2$ and so on. It is clear that $t_i; i = 1, 2, \ldots, n$, are unknown and time-steps are not essentially equidistant.

To start the solution process suppose $u_{j,i}$ and s_i stand for finite difference approximations of $u(x_j, t_i)$ and $s(t_i)$, respectively.

We start with the following approximation to find Δt_0. Suppose that,

$$\frac{ds}{dt} \simeq \frac{\Delta x}{\Delta t}. \tag{4.1}$$

Using the following equation at $t = t_1$,

$$\frac{ds}{dt} = F(s(t), u(s(t), t), \gamma), \quad 0 < t < T_f, \tag{4.2}$$

We can write

$$\frac{\Delta x}{\Delta t_i} \simeq F(s_i, u_i^i, \gamma) = F_i; i = 0, 1, ..., n, \tag{4.3}$$

At $t_0 = 0$, we conclude that

$$\frac{\Delta x}{\Delta t_0} = F_0. \tag{4.4}$$

F_0 is known, so Δt_0 is determined and we find t_1 as follows

$$\Delta t_0 = t_1 - t_0 = t_1 - 0 = t_1. \tag{4.5}$$

For simplicity, we suppose $D = D(t)$ and $p(u) = u$. Using these assumptions, one may find the descretize from of equation (2.8)-(2.13) as follows

$$\begin{aligned} \frac{u_{i,n+1} - u_{i,n}}{\Delta t_n} &= \frac{D_{n+1}}{s_{n+1}^2}\left[\frac{u_{i+1,n+1} - 2u_{i,n+1} + u_{i-1,n+1}}{\Delta x^2}\right] \\ &+ \frac{x_i F_n}{s_{n+1}} \frac{u_{i+1,n+1} - u_{i-1,n+1}}{2\Delta x} + u_i^{n+1} + f(x_i, t_n), \end{aligned} \tag{4.6}$$

$$\frac{\alpha_{11}}{s_{n+1}}\frac{u_{i+1,n+1}-u_{i-1,n+1}}{2\Delta x}+\alpha_{12}u_0^{n+1}=g(t_{n+1}), \tag{4.7}$$

$$\frac{\alpha_{21}}{sn+1}\left(u_{M+1,n+1}-u_{M-1,n+1}\right)+\alpha_{22}u_M^{n+1}=h(t_{n+1}), \tag{4.8}$$

where $D_n = D(t_n)$.

It should be noted that the auxiliary points $u_{M+1,n+1}$ and $u_{M-1,n+1}$ enter the calculations through relations (4.7) and (4.8), and we can obtain them from equations (4.7) and (4.8) and put them into equation (4.6) or use all the relations at the same time. In the second case, we have the following system of equations

$$\begin{aligned}&\left[\frac{\Delta t_n}{\Delta x^2}\frac{D_{n+1}}{s_{n+1}^2}+\frac{x_i}{2\Delta x}\frac{F_n}{s_{n+1}}\right]u_{i+1}^{n+1}+\left[-\frac{2\Delta t_n}{\Delta x^2}\frac{D_{n+1}}{s_{n+1}^2}+\Delta t_n\right]u_i^{n+1}\\&+\left[\frac{\Delta t_n}{\Delta x^2}\frac{D_{n+1}}{s_{n+1}^2}-\frac{x_i}{2\Delta x}\frac{\Delta t_n}{2\Delta x}\right]u_{i-1}^{n+1}=-u_i^n-f(x_i,t_{n+1}),\end{aligned} \tag{4.9}$$

$$\left[\frac{\alpha_{11}}{s_{n+1}}\frac{1}{2\Delta x}\right]u_{1,n+1}-\alpha_{12}u_0^{n+1}-\left[\frac{\alpha_{11}}{s_{n+1}}\frac{1}{2\Delta x}\right]u_{-1,n+1}=g(t_{n+1}), \tag{4.10}$$

$$\left[\frac{\alpha_{21}}{s_{n+1}}\frac{1}{2\Delta x}\right]u_{M+1,n+1}-\alpha_{22}u_M^{n+1}-\left[\frac{\alpha_{21}}{s_{n+1}}\frac{1}{2\Delta x}\right]u_{M-1,n+1}=h(t_{n+1}), \tag{4.11}$$

$$\Delta t_{n+1}=\frac{\Delta x}{F_{n+1}}. \tag{4.12}$$

In the foregoing equations, using (4.8) to (4.11), one may find $u_{n+1}^i; i = 0, 1, ..., n + 1$, and using (4.12), the value of Δt_{n+1} is obtained. Δt_{n+1} is used to find the value of t_{n+2} and this process can continue to find all unknowns.

5. Numerical Results

To show the ability and robustness of the numerical method presented in the previous section, in this section, we investigate two numerical examples. For the specific cases considered in this section, the convergence and stability properties may be found in [12, 13]. The numerical results are obtained using MATLAB and run on a Laptop equipped

with a Core i5 CPU and 8GB of RAM. To compare the numerical and exact solutions of the proposed test problems, we use the following L_∞-error norms.

$$\begin{aligned} e_u(s,\tau) &= \max_{1\le n\le N} \|U^n - u^n\|_\infty, \\ e_s(\tau) &= \max_{1\le n\le N} \|s_a^n - s^n\|_\infty. \end{aligned} \tag{5.1}$$

Example 1.

In problem (2.8)-(2.13), let $T_f = 1$, and

$$D = 1, \quad \rho(u) = u, \quad F = u^2, \quad f = -\mathrm{e}^{t-x}, \quad h = 0, \quad g = -2\,\mathrm{e}^t,$$
$$a_{11} = a_{12} = a_{21} = a_{22} = 1.$$

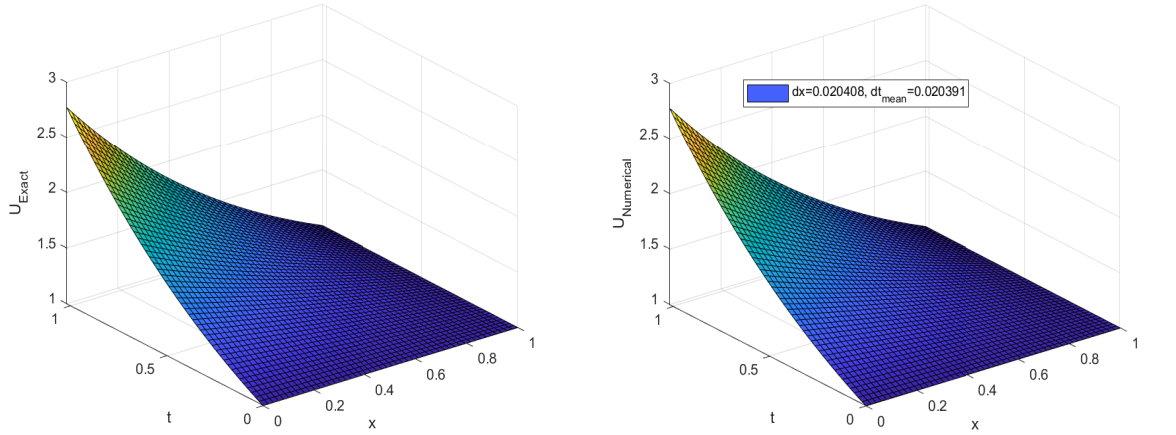
With these assumptions, one may find that the exact solution is as follows

$$\begin{aligned} u(x,t) &= \mathrm{e}^{t-x}, \\ s(t) &= t, \end{aligned}$$

Table 1 shows the L_∞-error norms between the exact and numerical results for u and s with different values of Δx. In Figure 1, we demonstrate the graph of the exact and numerical solutions using $M = 50$. Note that the time steps are variable.

TABLE 1. The L_∞-norm errors for Example 1.

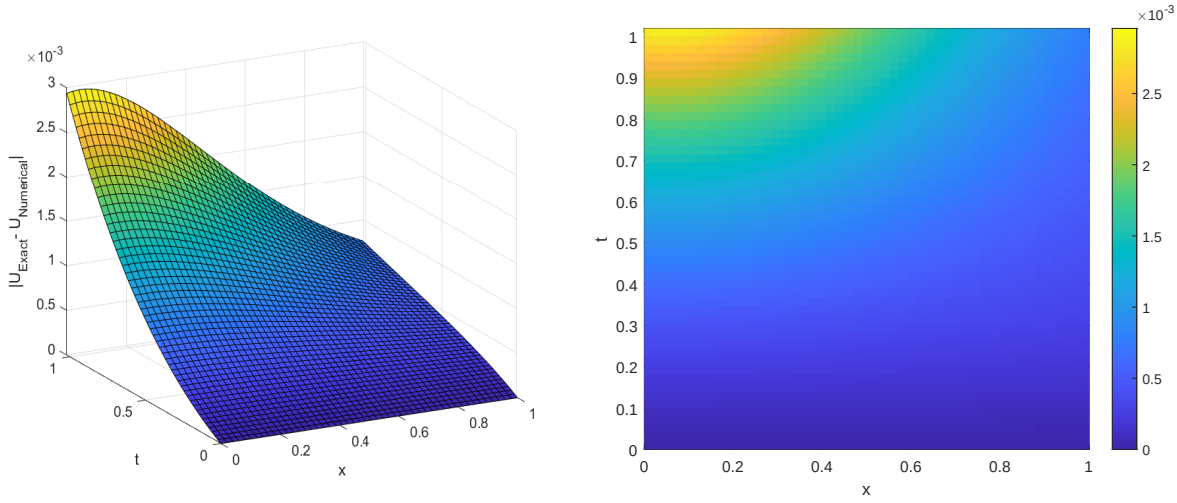
Δx	$e_u(dx, dt)$	$e_s(dt)$
1/8	0.0224	0.0073
1/16	0.0100	0.0029
1/32	0.0047	0.0013
1/64	0.0023	0.0006
1/128	0.0011	0.0003



(A) The exact solution of u (B) The numerical solution of u using $M = 50$

FIGURE 1. The numerical and exact solutions for Example 1.

In Figure 2, one may find the difference between the exact and numerical solutions of u.



(A) The difference between exact and numerical solutions of u. (B) The pseudocolor of the difference between the exact and the numerical solutions of u.

FIGURE 2. The difference between the numerical and exact solutions of u when $(x, t) \in (0, 1) \times (0, 1)$ in Example 1.

The exact and computed $s(t)$ are shown in Figure 3.

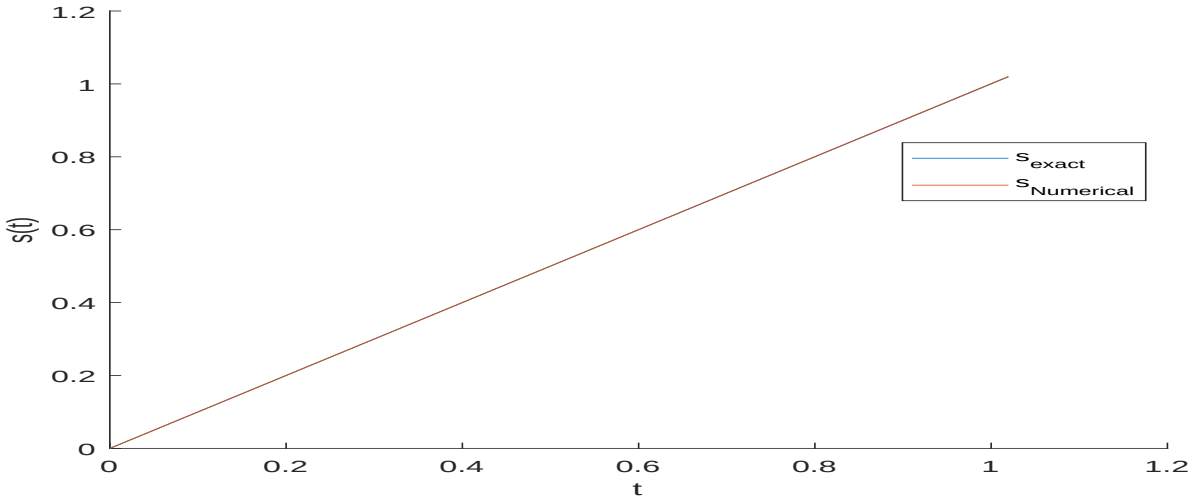
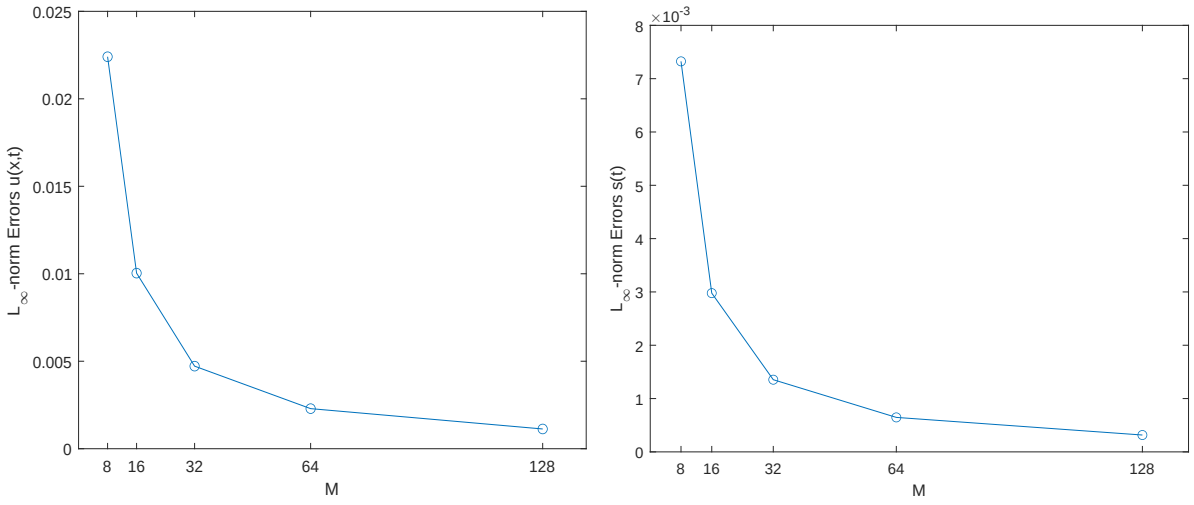


FIGURE 3. The exact and numerical solutions of $s(t)$ in Example 1.

The behavior of L_∞-norm error of u and s for different values of M is shown in Figure 4. This figure shows that as the number of mesh points increases, the amount of errors decreases. In this example, the comparison between the numerical and exact solutions clearly demonstrates that the numerical results are in good agreement with the exact solutions.



(A) (B)

FIGURE 4. A: The L_∞-norm error for $u(x,t)$, B: The L_∞-norm error for moving boundary $s(t)$ in Example 1.

Example 2. As the second test problem, in problem (8)-(13), consider $T_f = 1$, and

$$
\begin{aligned}
&D = 1, \quad \rho(u) == u, \quad F = u^2, \quad f = \frac{\mathrm{e}^{t/2}\,\cos(x)}{2}, \\
&h = \frac{\mathrm{e}^{t/2}}{\sqrt{(\mathrm{e}^t-1)^2+1}} - \frac{\mathrm{e}^{t/2}\,(\mathrm{e}^t-1)}{\sqrt{(\mathrm{e}^t-1)^2+1}}, \\
&g = -2\,\mathrm{e}^{t/2}, \quad a_{11} = a_{12} = a_{21} = a_{22} = 1.
\end{aligned} \tag{5.2}
$$

The exact solution of Example 2 may be found as

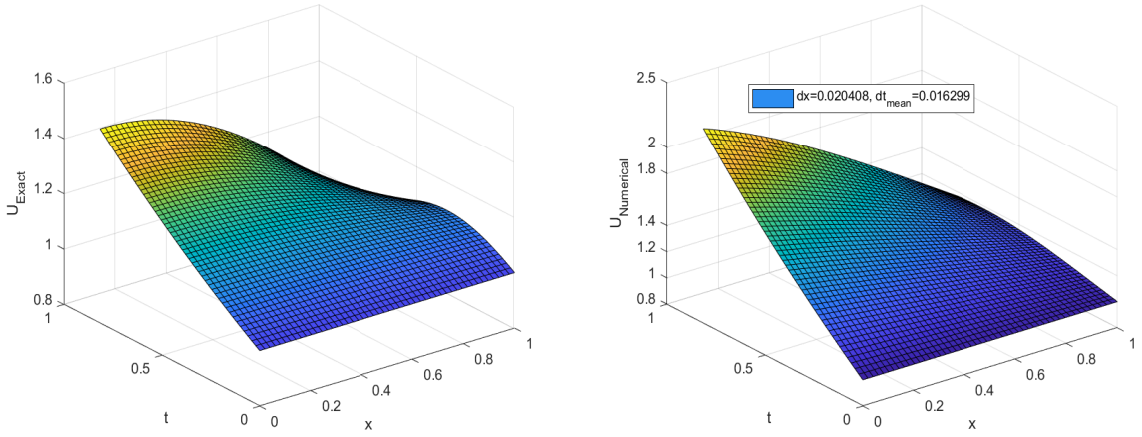
$$
\begin{aligned}
u(x,t) &= \mathrm{e}^{t/2}\,\cos(x)\,, \\
s(t) &= \arctan\left(\mathrm{e}^t - 1\right).
\end{aligned}
$$

In Table 2, we demonstrate the L_∞-error norms between the exact and numerical results for u and s with different values of Δx.

TABLE 2. The L_∞-norm errors for Example 2.

Δx	$e_u(dx, dt)$	$e_s(dt)$
1/8	0.9023	0.1643
1/16	0.8299	0.1362
1/32	0.7982	0.1246
1/64	0.7833	0.1193
1/128	0.7761	0.1168

Figure 5 shows the graph of the exact and numerical solutions for u using $M = 50$.



(A) The exact solution of u (B) The numerical solution of u

FIGURE 5. The numerical and exact solutions for Example 2 with $M = 50$.

A difference between the exact and numerical solutions for u is shown in Figure 5. Figure 6 shows the graph of the exact and computed $s(t)$ when $M = 50$.

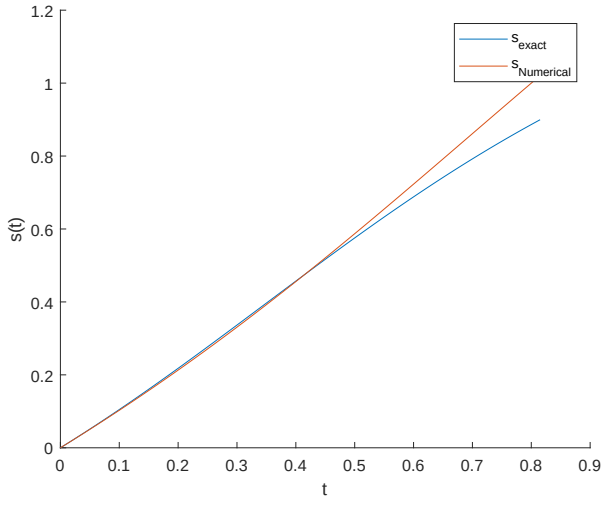
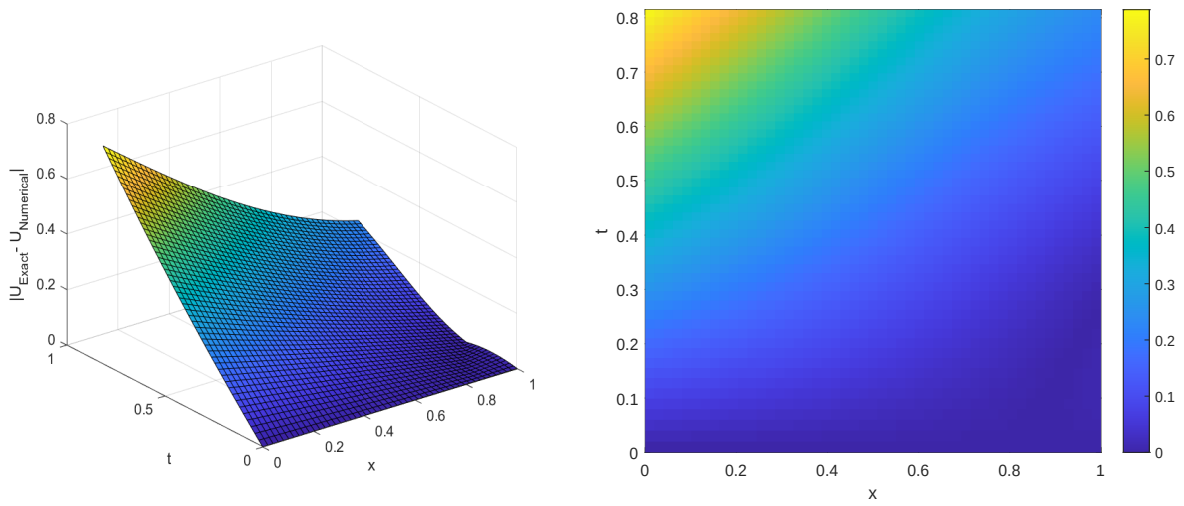


FIGURE 6. The exact and numerical solutions of $s(t)$ in Example 2 with $M = 50$.

Figure 6 shows the graph of the exact and computed $s(t)$ when $M = 50$.

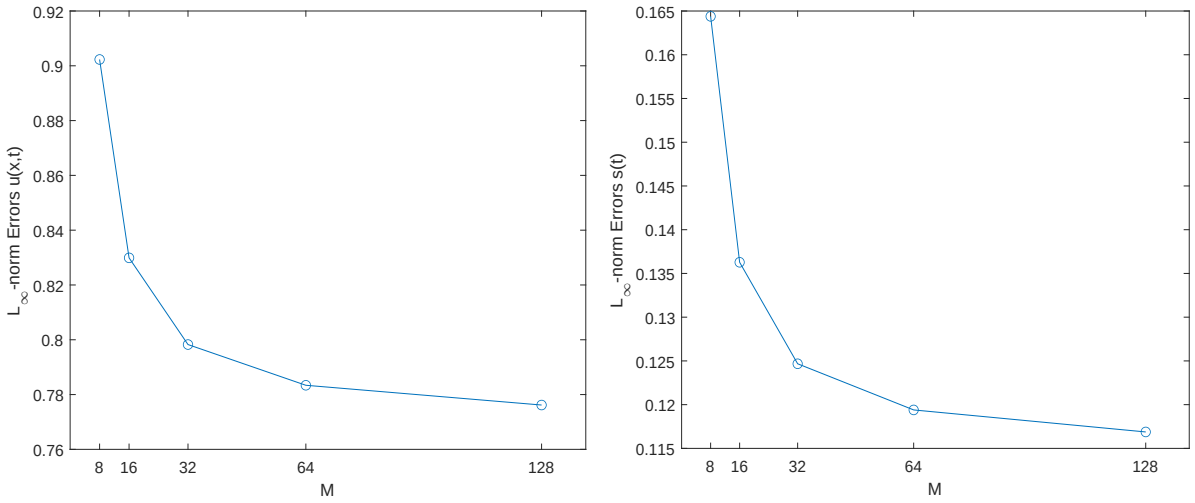


(A) The difference between exact and numerical solutions of u.

(B) The pseudocolor of the difference between the exact and the numerical solutions of u.

FIGURE 7. The difference between the numerical and exact solutions of u when $(x,t) \in (0,1) \times (0,1)$ and $m = 50$ in Example 2.

In Figure 8, the behaviors of L_∞-norm error of u and s for different values of M are shown. This Figure shows that increasing the number of spatial mesh points decreases the amount of numerical solution errors. In this example, as well as Example 1, the comparison between the numerical and exact solutions are well shows that the numerical results are in good agreement with the exact solutions.



(A) (B)

FIGURE 8. A: The L_∞-norm error for $u(x,t)$, B: The L_∞-norm error for moving boundary $s(t)$ in Example 2.

6. Conclusion

In this article, we investigate a one-dimensional parabolic moving boundary problem with variable coefficients, which models a wide range of physical phenomena. By employing a front-fixing transformation, the original problem is reformulated on a fixed spatial domain as a nonlinear partial differential equation. We establish the uniqueness of the solution to this transformed problem. To determine the unknown moving boundary and solve the system numerically, we develop a variable time-step finite difference scheme. Numerical experiments on two test problems are presented to validate the accuracy and robustness of the proposed method. Future work will focus on convergence and stability analysis, extending the approach to two- and three-dimensional moving boundary models, and exploring alternative numerical techniques suitable for such multidimensional problems.

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