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ON z°-IDEALS AND ANNIHILATOR IDEALS

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Abstract. For $a \in R$, let P_a denote the intersection of all minimal prime ideals of R containing a. An ideal I of a ring R is called a z°-ideal if $P_a \subseteq I$ for all $a \in I$. In this paper, we first investigate the class of z°-ideals in non-commutative rings. We provide characterizations of z°-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings, and semiprime rings. Next, we explore new properties of the lattice $rAnn(id(R))$, the set of right annihilator ideals of R. We prove that $rAnn(id(R))$ forms an algebraic frame when R is semiprime and a coherent frame when R is a reduced ring. Furthermore, we characterize the smallest (resp., largest) right annihilator ideal containing (resp., contained in) an ideal I of an SA-ring R.

1. Introduction

Throughout this paper, R denotes a nonzero associative ring with identity. The concept of z°-ideals was first introduced in commutative rings. An ideal of a ring R is called a z°-ideal if $P_a \subseteq I$, for every $a \in I$, where P_a is the intersection of all minimal prime ideals containing a. The study of z°-ideals has been pursued by many authors under different names. They were first investigated in [30] in the context of Baer rings under the name Baer ideals. In [11], the author referred to them as z-ideals. In [14], they were studied under the name pseudo-normal ideals, while the author of [16] called them B-ideals. In

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[5], they were referred to as ζ-ideals. Additionally, these ideals have been examined in Riesz spaces, f-rings, and frames under the name d-ideals, as seen in [12] (Section 3.3), [27] (Section 1), [17], [23], [24], and [18]. Due to their similarity to z-ideals in $C(X)$ (see [20]), the authors in [1, 2, 3, 6, 7, 8, 9, 10, 32] referred to them as z°-ideals.

In this paper, we extend the definition of z°-ideals from commutative rings to arbitrary rings. We recall in Section 2 the necessary background, and we fix notations. In Section 3, we investigate some properties of z°-ideals in semiprime rings. Notably, we observe that understanding z°-ideals in a ring R can be reduced to studying them in semiprime rings. We demonstrate that the class of z°-ideals includes the class of d-ideals. Furthermore, we provide examples of z°-ideals that are not d-ideals.

In Section 4, we characterize z°-ideals in various ring extensions, including 2-by-2 generalized triangular matrix rings, as well as full and upper triangular matrix rings.

Section 5 focuses on the lattice of right annihilator ideals. In Theorem 5.3, we prove that when R is a semiprime ring, the lattice of right annihilator ideals of R ($rAnn(id(R))$) is a frame. For a reduced ring R, it is shown that $rAnn(id(R))$ is a coherent frame (Theorem 5.6). Additionally, in Theorem 5.10, we provide a characterization of the largest (resp., smallest) annihilator ideal containing (resp., contained in) an ideal I of a right SA-ring R. Several corollaries and examples illustrating this result are also presented.

2. Background and Notation

2.1. **Rings.** For any subset S of R, l(S) and r(S) denote the left annihilator and the right annihilator of a subset S in R, respectively. An idempotent e of R is a left(right) semicentral idempotent if $Re = eRe (eR = eRe)$. The ring of n-by-n (upper triangular) matrices over R is denoted by $(\mathbf{T_n(R)})\mathbf{M_n(R)}$. A ring R is called a *right Ikeda– Nakayama* (for short, a right IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators; that is, if $l(I) + l(J) = l(I \cap J)$, for all right ideals I, J of R, see [15]. Recall from [13] that a ring R is called *right SA* if for each two ideals I, J of R, there exists an ideal K of R such that $r(I) + r(J) = r(K)$, (see also [21]).

2.2. **Algebraic frame.** Our reference for frames is [29]. A frame is a complete lattice L satisfying the distributively law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b : a \in A\},$$

for any subset$A \subseteq L$ and $b \in L$.

An element $a \in L$ is *compact*, if for any $X \subseteq L$, $a \leq \bigvee X$ implies that there is a finite subset F of X such that $a \leq \bigvee F$. We denote by $\mathfrak{k}(L)$, the set of all compact elements of L. If every element of L is the join of compact elements below it, then L is said to be algebraic. If $a \wedge b \in \mathfrak{k}(L)$ for every $a, b \in \mathfrak{k}(L)$, then L is said to have the *finite intersection property*, throughout abbreviated as FIP. If the top element of L (which we shall denote by 1) is compact and L has FIP, then L is called coherent.

3. Preliminary results and examples of z°-ideals

For any subset A of R, let P_A denote the intersection of all minimal prime ideals of R that contain A. When $A = \{a\}$ consists of a single element, we write P_a instead of P_A. Similar to the commutative case, an ideal I of a ring R is called a z°-ideal if $P_a \subseteq I$ for every $a \in I$. By definition, every minimal prime ideal of R is a z°-ideal and the intersection of any collection of z°-ideals is a z°-ideal. Consequently, the prime radical ideal of R, denoted as $P(R)$ is the smallest z°-ideal of R. This implies that the structure of z°-ideals of R is equivalent to that of $R/P(R)$. Thus, for the study of z°-ideals of a ring R, we may assume that $P(R) = (0)$, meaning that R can be considered a semiprime ring. We begin by recalling some well-known results from commutative rings and extend them to the non-commutative setting.

Example 3.1. (1) If I is a non-zero ideal (left ideal) in a semiprime ring R, then $l(I)$ is a z°-ideal. Since, we observe that $l(I) = \bigcap_{P \in \mathbf{Min(R)}, \mathbf{P} \not\supseteq \mathbf{I}} P$, by [25, Lemma 11.40]. This implies Re (resp., eR) is a z°-ideal, when e is a right (resp., left) semicentral idempotent. Since, $er = ere$, for each $r \in R$ and hence $er(1-e) = 0$. This shows $eR(1-e) = 0$. Hence $Re = l((1-e)R) = l(R(1-e))$.

(2) Every ideal in an strongly regular ring R (i.e., a ring R in which for each $a \in R$, there exists $b \in R$ such that $a = a^2b$) is an intersection of minimal prime ideals and hence, it is a z°-ideal. Since, every ideal in an strongly regular ring R is an intersection of minimal prime ideals.

(3) Let R be a reduced ring. Then for every minimal left ideal I of R, there is an idempotent $e \in R$ such that $I = Re = l((1-e)R) = l(R(1-e))$, by [25, Lemma 10.22], and hence IR is a z°-ideal, by Part (1).

Mason in [28] defined an ideal I of a reduced ring R as a d-ideal if for each $a \in I$, $r(l(a)) \subseteq I$. We extend this definition to arbitrary rings. Specifically, an ideal I of a ring R is called a right d-ideal if $r(l(RaR)) \subseteq I$ for each $a \in I$. Since in a reduced ring R, we have $rl(RaR) = r(l(a))$, this provides a natural extension of Mason's definition. We observe that the class of d-ideals and z°-ideals in a non-semiprime ring can be distinct. Consider the ring $\mathbb{Z}_{12}$ (ring of integers modulo 12). Then, it is easy to see that the ideal $< \overline{4} >$ is a d-ideal but not a z°-ideal. Since, $P_{\overline{4}} =< \overline{2} > \not\subseteq < \overline{4} >$.

Proposition 3.2. For a ring R the following statements are equivalent.

(1) For each $a \in R$, $P_a \subseteq rl(RaR)$.
(2) The ring R is semiprime.
(3) For $a, b \in R$, $P_b \subseteq P_a$ implies $l(RaR) \subseteq l(RbR)$.
(4) Every right d-ideal is a z°-ideal.
(5) Every right annihilator ideal of R is a z°-ideal.

Proof. (1)⇒(2) We have $P_\circ \subseteq r(l(0)) = 0$. This says the intersection of all minimal prime ideals of R is zero, i.e., R is a semiprime ring.

(2)⇒(3) Let $P_b \subseteq P_a$ and $x \in l(RaR)$. Then $xRaR = 0$ and hence $P_{xRbR} \subseteq P_{xRaR} = P_0 = 0$. This implies $xRbR = 0$. Thus $x \in l(RbR)$. Therefore, $l(RaR) \subseteq l(RbR)$.

(3)⇒(4) Let I be a right d-ideal, $a \in I$ and $b \in P_a$. Then $P_b \subseteq P_a$. By hypothesis, $l(RaR) \subseteq l(RbR)$. This implies $b \in r(l(RbR)) \subseteq rl(RaR) \subseteq I$.

(4)⇒(5) Suppose that I is a right annihilator and $a \in I$. Then $r(l(RaR)) \subseteq I$. It is easy to see that $r(l(RaR))$ is a right d-ideal, and so by hypothesis, is a z°-ideal. Thus, $a \in rl(RaR)$ implies $P_a \subseteq rl(RaR) \subseteq I$. That is I is a z°-ideal.

(5)⇒(1) For each $a \in R$, $r(l(RaR))$ is a z°-ideal, by hypothesis. Thus, $a \in rl(RaR)$ implies that $P_a \subseteq rl(RaR)$. □

Corollary 3.3. The class of z°-ideals and right d-ideals in a ring R coincide if and only if for every $a \in R$, $r(l(RaR)) = P_a$.

Proof. ⇒ By proposition 3.2, for each $a \in R$, we have $P_a \subseteq rl(RaR)$. Now, by hypothesis, P_a is a right d-ideal and $a \in P_a$, it follows that $r(l(RaR)) \subseteq P_a$, so we are done.

⇐ Again, by Proposition 3.2, every right d-ideal is a z°-ideal. Now, assume I is a z°-ideal and $a \in I$. Then, by definition, $P_a \subseteq I$. By hypothesis, $r(l(RaR)) = P_a$, which implies $r(l(RaR)) \subseteq I$, i.e., I is a right d-ideal. This completes the proof. □

Corollary 3.4. Let R be a reduced ring.

(1) For each $a \in R$, $P_a = r(l(RaR)) = r(l(a))$.

(2) The class of z°-ideals and right d-ideals coincide.

(3) A principal ideal RaR is a right annihilator ideal if and only if it is a z°-ideal.

Proof. (1) The second equality is evident. By Proposition 3.2, we have $P_a \subseteq rl(RaR)$. Now, suppose $x \in rl(RaR)$ and let $P \in \mathbf{Min(R)}$ with $a \in P$. Then $l(RaR)x = 0$ (i.e., $l(RaR)RxR = 0$) and by [22, Lemma 3.1], we have $l(RaR) = l(a) \not\subseteq P$. These conditions imply that $x \in P$, and hence $x \in P_a$.

(2) Follows from Part (1) and Corollary 3.3.

(3) If RaR is a right annihilator ideal, then $r(l(RaR)) = RaR$. By Part (1), we have $P_a = r(l(RaR))$, which implies $P_a = RaR$, meaning that RaR is a z°-ideal. Conversely, suppose RaR is a z°-ideal. Since $a \in RaR$, it follows that $P_a \subseteq RaR$. Again, by Part (1), we have $P_a = r(l(RaR))$, and thus $RaR = r(l(RaR))$. □

Lemma 3.5. Let I be an ideal (right ideal) of a semiprime ring R, $a \in R$ and $l(I) \subseteq l(a)$. Then, for each $y \in R$, $l(Iy) \subseteq l(ay)$.

Proof. Let $x \in l(Iy)$. Then $xIy = 0$. Thus $RxIy = 0$. This implies $(RyRxI)^2 = RyRxIRyRxI = RyRxIyRxI = 0$. By semiprime hypothesis, $RyRxI = 0$. This says $RyRx \subseteq l(I) \subseteq l(a)$. Hence, $RyRxa = 0$. This implies $(RxaRy)^2 = 0$, and hence by hypothesis, $RxaRy = 0$, i.e., $xay = 0$. Therefore, $x \in l(ay)$. □

Proposition 3.6. Let I and P be ideals of a reduced ring R and P be a prime ideal. If $I \cap P$ is a z°-ideals, then either I or P is a z°-ideal. In particular, if P and Q are prime ideals of R which are not in a chain and $P \cap Q$ is a z°-ideal, then both P and Q are z°-ideals.

Proof. If $I \subseteq P$, then $I \cap P = I$ is a z°-ideal. Suppose $I \not\subseteq P$, $a \in P$ and $l(RaR) \subseteq l(RxR)$. Take $y \in I \setminus P$. By Lemma 3.5, $l(RaRy) \subseteq l(RxRy)$. Since $RaRy \subseteq I \cap P$ and $I \cap P$ is a z°-ideal, it follows from Corollary 3.4 that $RxRy \subseteq I \cap P$. Consequently, $RxRy \subseteq P$, which implies $x \in P$. The proof of the second part of the proposition follows similarly. □

Example 3.7. Let $R = \begin{pmatrix} \mathbb{Z}_6 & \mathbb{Z}_6 \\ 0 & \mathbb{Z}_6 \end{pmatrix}$. Then, by Theorem 4.9, $I = \begin{pmatrix} < \overline{3} > & \mathbb{Z}_6 \\ 0 & < \overline{2} > \end{pmatrix}$ is a z°-ideal in R, where $< \overline{2} >$ and $< \overline{3} >$ are ideals generated by $\overline{2}$ and $\overline{3}$, respectively. However,

the ideal I is not a right d-ideal. For, $A = \begin{pmatrix} \overline{3} & \overline{1} \\ 0 & 0 \end{pmatrix} \in I$ and we have $RA = \begin{pmatrix} < \overline{3} > & \mathbb{Z}_6 \\ 0 & 0 \end{pmatrix}$. Thus $l(RAR) = l(RA) = \begin{pmatrix} 0 & \mathbb{Z}_6 \\ 0 & \mathbb{Z}_6 \end{pmatrix}$ and this implies that $r(l(RAR)) = \begin{pmatrix} \mathbb{Z}_6 & \mathbb{Z}_6 \\ 0 & 0 \end{pmatrix} \not\subseteq I$.

4. z°-IDEALS IN EXTENSION RINGS

In this section, we determine z°-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings. First, we need to extend the definition of z°-ideals as the following.

For $n \in \mathbb{N}$, we call an ideal I of R a z_n°-ideal if for every finite subset F of I with at most n elements, we have $P_F \subseteq I$. Clearly, z_1°-ideals precisely are the z°-ideals. Moreover, an ideal I of R is a sz°-ideal (see [4]) if and only if I is a z_n°-ideal for each $n \in \mathbb{N}$. For $n > 1$, every z_n°-ideal is also a z_{n-1}°-ideal, and hence a z°-ideal. However, the converse does not necessarily hold. Indeed, there exists an example of a z°-ideal (hence z_1°-ideal) that is not an sz°-ideal and consequently, not a z_n°-ideal for some $n > 1$, see [4, Example 4.2].

Lemma 4.1. Let $A = [a_{ij}] \in \mathbf{M_n(R)}$.
Then $P_A = \mathbf{M_n(P_B)}$, where $B = \{a_{ij} : 1 \leq i, j \leq n\}$.

Proof. By [24, Theorem 3.1], every ideal of $\mathbf{M_n(R)}$ is of the form $M_n(I)$ for some ideal I of R. Moreover, $M_n(I)$ is a minimal prime ideal of $\mathbf{M_n(R)}$ if and only if I is a minimal prime ideal of R. By definition, we have:

$$P_A = \bigcap_{A \in \mathbf{M_n(P)} \in \mathbf{Min(M_n(R))}} \mathbf{M_n(P)} = \mathbf{M_n}\Big(\bigcap_{\mathbf{A} \in \mathbf{M_n(P)} \in \mathbf{Min(M_n(R))}} \mathbf{P} \Big) =$$

$$\mathbf{M_n}\Big(\bigcap_{\mathbf{a_{ij}} \in \mathbf{P} \in \mathbf{Min(R)}, \mathbf{1 \leq i,j \leq n}} \mathbf{P} \Big) = \mathbf{M_n(P_{\{a_{ij} : 1 \leq i,j \leq n\}})} = \mathbf{M_n(P_B)}.$$

□

Theorem 4.2. An ideal J of $\mathbf{M_n(R)}$ is a z°-ideal if and only if $J = M_n(I)$ for some $z_{n^2}^\circ$-ideal I of R.

Proof. $\Rightarrow$ Let J be a z°-ideal of $\mathbf{M_n(R)}$. By [24, Theorem 3.1], $J = M_n(I)$ for some ideal I of R. We claim that I is a $z_{n^2}^\circ$-ideal of R. Let F be a subset of I with at most n^2 elements.

Without loss of generality, we assume $F = \{a_{11}, a_{12}, ...a_{1n}, a_{21}, a_{22}, ..., a_{2n}, ..., a_{n1}, a_{n2}, ..., a_{nn}\}$. Put $A = [a_{ij}]$, so that $A \in \mathbf{M_n(R)}$. By Lemma 4.1, we obtain $P_A = \mathbf{M_n(P_F)}$. By hypothesis, $P_A \subseteq J$, which implies $P_F \subseteq I$. Hence, I is a $z^\circ_{n^2}$-ideal of R.

$\Leftarrow$ Let I be a $z^\circ_{n^2}$-ideal of R and let $A = [a_{ij}] \in J = M_n(I)$. Define $F = \{a_{ij} : 1 \leq i, j \leq n\}$. By Lemma 4.1, $P_A = \mathbf{M_n(P_F)}$. Since $F \subseteq I$ and contains at most n^2 elements and I is a $z^\circ_{n^2}$-ideal, we have $P_F \subseteq I$, which implies $P_A = \mathbf{M_n(P_F)} \subseteq M_n(I) = J$. □

Example 4.3. Consider the ring $R = \mathbb{Z}_8$. Then, it is easy to see that $I =< \overline{2} >$ is a z°_4-ideal of R. Now, by Theorem 4.2, $M_2(< \overline{2} >)$ is a z°-ideal of $M_2(\mathbb{Z}_8)$.

By [33, Theorem 3.2], every ideal of $\mathbf{T_n(R)}$ is of the following form:

$$I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}, J_{ik} \subseteq J_{ik+1} \quad \text{and } J_{i+1k} \subseteq J_{ik}.$$

It is easy to see that I is a prime ideal of $\mathbf{T_n(R)}$ if and only if all $J_{ij} = R$ $(1 \leq i, j \leq n)$ except one of J_{ii} which is a prime ideal of R.

Lemma 4.4. Let $A = [a_{ij}] \in \mathbf{T_n(R)}$. Then

$$P_A = \begin{pmatrix} P_{a_{11}} & R & R & . & . & . & R \\ 0 & P_{a_{22}} & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & P_{a_{nn}} \end{pmatrix}$$

Proof. Let $A = [a_{ij}] \in \mathbf{T_n}(\mathbf{R})$. Then

$$P_A = (\bigcap_{a_{11}\in P\in \mathbf{Min}(\mathbf{R})} \begin{pmatrix} P & R & R & . & . & . & R \\ 0 & R & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & R \end{pmatrix}) \bigcap (\bigcap_{a_{22}\in P\in \mathbf{Min}(\mathbf{R})} \begin{pmatrix} R & R & R & . & . & . & R \\ 0 & P & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & R \end{pmatrix}) \ldots$$

$$\bigcap (\bigcap_{a_{nn}\in P\in \mathbf{Min}(\mathbf{R})} \begin{pmatrix} R & R & R & . & . & . & R \\ 0 & R & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & P \end{pmatrix}) = \begin{pmatrix} \bigcap_{a_{11}\in P\in \mathbf{Min}(\mathbf{R})} P & R & R & . & . & . & R \\ 0 & R & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & R \end{pmatrix}$$

$$\bigcap \begin{pmatrix} R & R & R & . & . & . & R \\ 0 & \bigcap_{a_{22}\in P\in \mathbf{Min}(\mathbf{R})} P & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & R \end{pmatrix} \ldots \bigcap \begin{pmatrix} R & R & R & . & . & . & R \\ 0 & R & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & \bigcap_{a_{nn}\in P\in \mathbf{Min}(\mathbf{R})} P \end{pmatrix} =$$

$$\begin{pmatrix} P_{a_{11}} & R & R & . & . & . & R \\ 0 & P_{a_{22}} & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & P_{a_{nn}} \end{pmatrix}.$$

□

Theorem 4.5. The ideal $I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}$ of $\mathbf{T_n(R)}$ is a z°-ideal if and only if each J_{ii} $(1 \leq i \leq n)$ is a z°-ideal and $J_{ij} = R$ for all $j > i$.

Proof. $\Rightarrow$ First, we note that $[0] \in I$. By Lemma 4.4, this implies $P_{[0]} = [P_{ij}] \subseteq I$, where $P_{ii} = P_0$ for all $1 \leq i \leq n$ and $P_{ij} = R$ whenever $j > i$. Consequently, we have $J_{ij} = R$ for all $j > i$. Next, we claim that J_{ii} (for $1 \leq i \leq n$) is a z°-ideal. Fix $1 \leq i \leq n$ and let $a \in J_{ii}$. Define the matrix $A = [a_{ij}]$ as follows:
a occupies the (i,i) position, $a_{jj} = 0$ for all $1 \leq j \neq i \leq n$, $a_{ij} = 0$ for $i > j$, and all other entries are 1.
Since $A \in I$, Lemma 4.4 gives us, $P_A = [P_{ij}]$, where $P_{ii} = P_a$, $P_{jj} = P_0$ for all $1 \leq j \neq i \leq n$, $P_{ij} = 0$ for $i > j$ and elsewhere is R . By hypothesis, $P_A \subseteq I$, which implies $P_a \subseteq J_{ii}$. Thus J_{ii} is a z°-ideal.

$\Leftarrow$ Let $A = [a_{ij}] \in I$. By hypothesis,

$$I = \begin{pmatrix} J_{11} & R & R & . & . & . & R \\ 0 & J_{22} & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & R \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}.$$

By Lemma 4.4,

$$P_A = \begin{pmatrix} P_{a_{11}} & R & R & . & . & . & R \\ 0 & P_{a_{22}} & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & P_{a_{nn}} \end{pmatrix}.$$

Since for each $1 \leq i \leq n$, J_{ii} is a z°-ideal, $P_{a_{ii}} \subseteq J_{ii}$. Hence $P_A \subseteq I$. □

Example 4.6. (1) By Theorem 4.5, the zero-ideal of $T_n(R)$ is not a z°-ideal, even when R is a semiprime ring.

(2) Let R be a semiprime ring. Then the zero ideal is a z°-ideal in R. Hence, by theorem 4.5, the following ideal is a z°-ideal in $\mathbf{T_n(R)}$.

$$\begin{pmatrix} 0 & R & R & . & . & . & R \\ 0 & 0 & R & . & . & . & R \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & R \\ 0 & 0 & . & . & . & 0 & 0 \end{pmatrix}.$$

We are including the following lemma for completeness since it is used in the next result.

Lemma 4.7. An ideal $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ is a prime ideal if and only if

(i) $N = M$.

(ii) $I = S$ and L is a prime ideal of R or $L = R$ and I is a prime ideal of S.

Proof. $\Rightarrow$ By [24, Proposition 1.17], $K = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is an ideal of T. Since $K^2 = 0 \subseteq J$, the hypothesis implies, $K \subseteq J$, which gives $N = M$. Again by [24, Proposition 1.17], $I_1 = \begin{pmatrix} I & M \\ 0 & R \end{pmatrix}$ and $I_2 = \begin{pmatrix} S & M \\ 0 & L \end{pmatrix}$ are ideals of T and $I_1I_2 = \begin{pmatrix} I & M \\ 0 & L \end{pmatrix}$=J. Thus, we must have either $I_1 \subseteq J$ or $I_2 \subseteq J$, which implies $I = S$ or $L = R$. Now, assume $I = S$. Let L_1, L_2 be two ideals of R such that $L_1L_2 \subseteq L$. Then, $H_1 = \begin{pmatrix} 0 & M \\ 0 & L_1 \end{pmatrix}$ and $H_2 = \begin{pmatrix} 0 & M \\ 0 & L_2 \end{pmatrix}$ are two ideals of T and $H_1H_2 \subseteq J$. Consequently, we must have either $H_1 \subseteq J$ or $H_2 \subseteq J$, which implies $L_1 \subseteq L$ or $L_2 \subseteq L$. Similarly, when $L = R$, we conclude that I is a prime ideal of R.

$\Leftarrow$ It is easy to see that whenever I and L are prime ideals of S and R, respectively, the ideals $\begin{pmatrix} S & M \\ 0 & L \end{pmatrix}$ and $\begin{pmatrix} I & M \\ 0 & R \end{pmatrix}$ are prime ideals of T. □

Lemma 4.8. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in T$. Then $P_A = \begin{pmatrix} P_{a_{11}} & M \\ 0 & P_{a_{22}} \end{pmatrix}$.

Proof. By Lemma 4.7, we have

$$P_A = \Big(\bigcap_{a_{11}\in P\in \mathbf{Min(S)}} \begin{pmatrix} P & M \\ 0 & R \end{pmatrix}\Big)\bigcap\Big(\bigcap_{a_{22}\in P\in \mathbf{Min(R)}} \begin{pmatrix} S & M \\ 0 & P \end{pmatrix}\Big) =$$

$$\begin{pmatrix} \bigcap_{a_{11}\in P\in \mathbf{Min(S)}} P & M \\ 0 & \bigcap_{a_{22}\in P\in \mathbf{Min(R)}} P \end{pmatrix} = \begin{pmatrix} P_{a_{11}} & M \\ 0 & P_{a_{22}} \end{pmatrix}.$$

□

The following result provides a characterization of z°-ideals in a 2-by-2 generalized triangular matrix ring.

Theorem 4.9. Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

Then J is a z°-ideal of T if and only if

(i) $N = M$.

(ii) Two ideals I, L are z°-ideals of S and R, respectively.

Proof. $\Rightarrow$ Let $n \in N$. Consider the matrix $A = \begin{pmatrix} 0_S & n \\ 0 & 0_R \end{pmatrix} \in J$. By hypothesi and Lemma 4.8, we have $P_A = \begin{pmatrix} P_0 & M \\ 0 & P_0 \end{pmatrix} \subseteq J$. This implies that $M = N$. Now, let $a \in I$ and $b \in L$. Consider the matrix $B = \begin{pmatrix} a & n \\ 0 & b \end{pmatrix} \in J$ for each $n \in M$. By assumption, we must have $P_B \subseteq J$, which implies that $P_a \subseteq I$ and $P_b \subseteq L$. This shows that I and L are z°-ideals of S and R, respectively.

$\Leftarrow$ Let $A = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \begin{pmatrix} I & M \\ 0 & L \end{pmatrix}$. Then $a \in I$ and $b \in L$. By hypothesis, $P_a \subseteq I$, $P_b \subseteq L$ and by Lemma 4.8, $P_A = \begin{pmatrix} P_a & M \\ 0 & P_b \end{pmatrix}$. Thus $P_A \subseteq J$. □

5. On the lattice of annihilator ideals

An ideal I of a ring R is a right (left) annihilator ideal if rl(I)=I (lr(I)=I); equivalently, $l(I) \subseteq l(x)$ $(r(I) \subseteq r(x))$ and $x \in R$, imply $x \in I$.

Put $rAnn(id(R)) = \{I : I \text{ is a right annihilator ideal of } R\}$. In [13] and [19] it is shown that $rAnn(id(R))$ is a complete lattice with the following operations.

$$I \vee J = r_R(l_R(I) \cap l_R(J)) \quad \text{and} \quad I \wedge J = I \cap J.$$

In this section, we try to investigate some new properties of this lattice.
In the following result some items are well-known. However, we bring their proofs.

Lemma 5.1. The following statements are equivalent.

(1) The ring R is semiprime.
(2) For every two ideals I, J of R, $l(IJ) = l(I \cap J)$ $(r(IJ) = r(I \cap J))$.
(3) For every two ideals I, J of R, $rl(IJ) = rl(I \cap J)$.

Proof. (1)⇒(2) Clearly $l(I \cap J) \subseteq l(IJ)$, since $IJ \subseteq I \cap J$. For the other inclusion, consider $r \in l(IJ)$ and $s \in I \cap J$. Then $RsRsR \subseteq IJ$ and $RrR \subseteq l(IJ)$ and hence we have $(RrRsR)^2 = RrRsRrRsR \subseteq RrRsRsR = 0$. By assumption, this shows that $RrRsR = 0$ and thus $rs = 0$. Therefore, $r \in l(I \cap J)$. So we are done.

(2)⇒(1) Let I be an ideal of R and $I^2 = 0$. Then we have

$$l(I) = l(I \cap I) = l(I^2) = R.$$

This implies that $I = 0$.

(2)⇒(3) Trivial.

(3)⇒(1) Let I be an ideal of R and $I^2 = 0$. Then, by hypothesis, $rl(I) = rl(I \cap I) = rl(I^2) = 0$. Thus, $l(I) = lrl(I) = R$, i.e., $I = 0$. □

Lemma 5.2. Let R be a semiprime. Then, for any family $\{I_\alpha : \alpha \in S\}$ of ideals of R and any right annihilator ideal J of R, we have

$$J \cap rl(\sum_{\alpha \in S} I_\alpha) = rl(\sum_{\alpha \in S} I_\alpha \cap J).$$

Proof. ⇒ Since J is a right annihilator ideal, we have

$$rl(\sum_{\alpha \in S} I_\alpha \cap J) \subseteq J \cap rl(\sum_{\alpha \in S} I_\alpha).^{(1)}$$

On the other hand, we have

$$\sum_{\alpha \in S} (J \cap I_\alpha) \subseteq rl(\sum_{\alpha \in S} J \cap I_\alpha).$$

This implies that for each $\alpha \in S$, we have $J \cap I_\alpha \subseteq rl(\sum_{\alpha\in S} J \cap I_\alpha)$. By hypothesis, this implies

$$J \cap I_\alpha \cap l(\sum_{\alpha\in S} J \cap I_\alpha) = 0, \quad \text{for each} \quad \alpha \in S,$$

and hence $J \cap l(\sum_{\alpha\in S} J \cap I_\alpha) \subseteq l(I_\alpha)$, for each $\alpha \in S$. Taking the intersection over all α we obtain:

$$J \cap l(\sum_{\alpha\in S} J \cap I_\alpha) \subseteq \bigcap_{\alpha\in S} l(I_\alpha) = l(\sum_{\alpha\in S} I_\alpha).$$

This leads to $J \cap l(\sum_{\alpha\in S} J \cap I_\alpha) \cap rl(\sum_{\alpha\in S} I_\alpha) = 0$. Thus,

$$J \cap rl(\sum_{\alpha\in S} I_\alpha) \subseteq rl(\sum_{\alpha\in S} J \cap I_\alpha).^{(2)}$$

Combining (1) and (2), we establish the required equality. □

Theorem 5.3. Let R be a semiprime. Then, $< rAnn(id(R)), \subseteq >$ is a frame.

Proof. Let J and the family $\{I_\alpha : \alpha \in S\}$ be right annihilator ideals. Then, by Lemma 5.2,

$$J \wedge (\vee_{\alpha\in S} I_\alpha) = J \cap rl(\sum_{\alpha\in S} I_\alpha) = rl(\sum_{\alpha\in S} J \cap I_\alpha) = \bigvee_{\alpha\in S} (J \wedge I_\alpha).$$

This shows that $rAnn(id(R))$ is a frame. □

Lemma 5.4. Let R be a semiprime ring. Then the compact elements of $rAnn(id(R))$ are precisely the ideals of the form $rl(Ra_1R) \vee ... \vee rl(Ra_nR)$, for some finitely many elements $a_i \in R$. Indeed,

$$\mathfrak{k}(rAnn(id(R))) = \{rl(Ra_1R) \vee ... \vee rl(Ra_nR) : a_1, ..., a_n \in R, n \in \mathbb{N}\}.$$

Proof. To show that each $rl(Ra_1R) \vee ... \vee rl(Ra_nR)$ is compact, we first show that each $rl(RaR)$ is compact. Let $\{I_\alpha : \alpha \in S\}$ be a directed collection of right annihilator ideals with $rl(RaR) \leq \bigvee_{\alpha\in S} I_\alpha$. Then $rl(RaR) \subseteq \bigcup_{\alpha\in S} I_\alpha$. Since $a \in rl(RaR)$, we have $a \in I_\alpha$, for some $\alpha \in S$. This implies $rl(RaR) \subseteq I_\alpha$, since I_α is a right annihilator ideal. Therefore, $rl(RaR)$ is compact, and hence $rl(Ra_1R) \vee ... \vee rl(Ra_nR)$ is compact. Now, let $K \in \mathfrak{k}(rAnn(id(R)))$. Since K is a right annihilator ideal, for each $a \in K$, $rl(RaR) \subseteq K$, hence $\bigvee_{a\in K} rl(RaR) \leq K$. On the other hand, $K \leq \sum_{a\in K} rl(RaR) \leq \bigvee_{a\in K} rl(RaR)$. Thus $K = \bigvee_{a\in K} rl(RaR)$. Since K is compact, there are finitely elements $a_1, ..., a_n \in R$ such that $K = \bigvee_{i=1}^{n} rl(Ra_iR)$. This completes our proof. □

If I is a right annihilator ideal, then $I = \bigvee_{a\in I} rl(RaR)$. This together with the above result imply that $rAnn(id(R))$ is an algebraic frame.

Lemma 5.5. Let R be a semiprime ring. Then R is a reduced ring if and only if for each $a, b \in R$, $l(RaRbR) = l(RabR)$.

Proof. $\Rightarrow$ Clearly $l(RaRbR) \subseteq l(RabR)$, since $RabR \subseteq RaRbR$. Now, assume $x \in l(RabR)$. Then $xab = 0$, which implies $(bxa)^2 = 0$. By the reduced hypothesis, we obtain $bxa = 0$ and so $b \in l(xaR) = r(xaR)$. Thus $xaRb = 0$, which shows that $xaRbR = 0$. This leads us to $x \in l(aRbR) = r(aRbR)$. Meaning that $aRbRx = 0$ and thus $RaRbRx = 0$. This implies $x \in r(RaRbR) = l(RaRbR)$.

$\Leftarrow$ Let $a \in R$ such that $a^2 = 0$. Then $l((RaR)^2) = l(Ra^2R) = R$, by hypothesis. Hence $(RaR)^2 = 0$. Since R is semiprime, we conclude that $RaR = 0$. Therefore, $a = 0$. □

The goal is to show that $rAnn(id(R))$ is a coherent frame. We thus need to have the meet of two compact elements. For that we apply Lemmas 5.4 and 5.5.

Theorem 5.6. Let R be a reduced ring. Then $rAnn(id(R))$ is a coherent frame.

Proof. We first note that $R = rl(R1R)$ (i.e., the top element) is compact in $rAnn(id(R))$, by Lemma 5.4. To see coherence, let $K_1, K_2 \in \mathfrak{k}(rAnn(id(R)))$ with, say,

$$K_1 = rl(Ra_1R) \vee ... \vee rl(Ra_kR) \quad \text{and} \quad K_2 = rl(Rb_1R) \vee ... \vee rl(Rb_nR).$$

Then Lemmas 5.1 and 5.5 imply

$$K_1 \wedge K_2 = rl(Ra_1R) \cap rl(Rb_1R) \vee ... \vee rl(Ra_1R) \cap rl(Rb_nR) \vee ... \vee$$

$$rl(Ra_kR) \cap rl(Rb_1R) \vee ... \vee rl(Ra_kR) \cap rl(Rb_nR) =$$

$$rl(Ra_1Rb_1R) \vee ... \vee rl(Ra_1Rb_nR) \vee ... \vee rl(Ra_kRb_1R) \vee ... \vee rl(Ra_kRb_nR) =$$

$$rl(Ra_1b_1R) \vee ... \vee rl(Ra_kb_nR).$$

Now by using Lemma 5.4, $K_1 \wedge K_2$ is compact. Therefore, $rAnn(id(R))$ is a coherent frame. □

Proposition 5.7. For each ideal I of R, the smallest right annihilator ideal containing I equals to the ideal

$$I_A = \bigvee_{a\in I} rl(RaR).$$

Proof. It is evident $rl(I)$ is a right annihilator ideal containing I. Now, let J be a right annihilator ideal containing I. Then $l(J) \subseteq l(I)$, and hence $rl(I) \subseteq rl(J)) = J$. Thus, $rl(I)$ is the smallest right annihilator ideal containing I. We denote it by I_A and we claim that

$$I_A = \bigvee_{a \in I} rl(RaR).$$

To see it, first note that $\bigvee_{a\in I} rl(RaR)$ is a right annihilator ideal, since $rAnn(id(R))$ is a complete lattice. Next, for each $a \in I$, $rl(RaR) \subseteq rl(I) = I_A$. Thus,

$$I \subseteq \sum_{a\in I} rl(RaR) \subseteq \bigvee_{a\in I} rl(RaR) \subseteq I_A.$$

Since I_A is the smallest right annihilator ideal containing I, the proof is complete. □

Lemma 5.8. Let I and J be two ideals of R.

(1) $I \subseteq J$ implies $I_A \subseteq J_A$.
(2) If R is a reduced ring, then $I \subseteq \sqrt{I} \subseteq I_A$. Hence $I_A = \sqrt{I}_A$.
(3) If R is a semiprime ring, then $(I \cap J)_A = I_A \cap J_A$.
(4) $(I + J)_A = (I_A + J_A)_A$.

Proof. (1) Trivial.

(2) It is enough to show that $\sqrt{I} \subseteq I_A$. Consider $x \in \sqrt{I}$ and $r \in l(I)$. Then $x^n \in I$ for some $n \in \mathbb{N}$ and hence $rx^n = 0$. By Lemmas 5.1 and 5.5 we have

$$0 = rl(Rrx^nR) = rl(RrRx^nR) = rl(RrR) \cap rl(Rx^nR) =$$

$$rl(RrR) \cap rl(RxR) = rl(RrRxR) = rl(RrxR).$$

This implies $l(RrxR) = R$ and hence $rx = 0$, i.e., $x \in rl(I) = I_A$.

(3) By Lemma 5.1, we have

$$I_A \cap J_A = rl(I) \cap rl(J) = rl(I \cap J) = (I \cap J)_A.$$

(4) We have,

$$(I + J)_A = rl(I + J) = rl(I) \cap rl(J) =$$

$$rlr(l(I)) \cap rlr(l(J)) =$$

$$rl(rl(I) + rl(J)) =$$

$$rl(I_A + J_A) = (I_A + J_A)_A.$$

□

Corollary 5.9. The following statements hold.

(1) For an ideal I of a semiprime ring R, $I_A = R$ if and only if I is an essential ideal.
(2) A maximal ideal M of a semiprime ring R is a right annihilator ideal if and only if it is generated by an idempotent.

Proof. (1) The equality $rl(I) = I_A = R$ is equivalent to the $l(I) = 0$. By [25, Corollary 14.2], $l(I) = 0$ if and only if I is an essential ideal.

(2) It is well-known that every maximal ideal M in a ring R is either essential or generated by an idempotent. Now, if M is a right annihilator ideal, then $M = r(l(M)) = M_A$. By Part (1), this is equivalent to the M being a non-essential ideal, i.e., M must be generated by an idempotent. □

In the next result, we provide a characterization of the largest (resp., smallest) annihilator ideal contained in (resp., consisting of) an ideal I of an SA-ring R. We denote the largest right annihilator ideal contained in I by I^A.

Theorem 5.10. For any ring R the following statements are equivalent.

(1) For each ideal I of R, the largest right annihilator ideal contained in I exists.
(2) The ring R is a right SA-ring.
(3) For each two ideals I and J of R, $(I + J)_A = I_A + J_A$.

Furthermore, if one of the above conditions satisfies on R, then for an ideal I of R,

$$I_A = \sum_{a \in I} rl(RaR) \quad ^\dagger \quad \text{and} \quad I^A = \sum_{rl(RaR) \subseteq I} rl(RaR) \quad ^\ddagger.$$

Proof. (1)⇒(2) Consider two right annihilator ideals I and J in R. By hypothesis, the largest right annihilator ideal contained in $I + J$, $(I + J)^A$, exists. Since I and J are right annihilator ideals contained in $I + J$, we have $I \subseteq (I + J)^A$ and $J \subseteq (I + J)^A$. This implies $I + J \subseteq (I + J)^A$ and hence we must have $I + J = (I + J)^A$, i.e., $I + J$ is a right annihilator ideal.

(2)⇒(3) Clearly $I_A + J_A \subseteq (I + J)_A$. To prove the reverse inclusion, we note that $I_A + J_A$ is a right annihilator ideal containing $I + J$, by hypothesis. Thus, $(I + J)_A \subseteq I_A + J_A$, so we are done.

(3)⇒(1) Clearly the zero ideal is a right annihilator ideal contained in any ideal I of R. Thus, the set of right annihilator ideals contained in I is non-empty. The hypothesis implies that for every two right annihilator ideals I and J of R, $I+J = I_A+J_A = (I+J)_A$. That is the sum of two right annihilator ideals in R is a right annihilator ideal. Put

$$I^A = \sum_{J\in rAnn(id(R)),\quad J\subseteq I} J.$$

By hypothesis, I^A is a right annihilator ideal and $I^A \subseteq I$. This shows that I^A is the largest right annihilator ideal contained in I.

Now, assume R is a right SA-ring. To prove the equality †, we note that for each ideal I of R we have,

$$I \subseteq \sum_{a\in I} rl(RaR) \subseteq \bigvee_{a\in I} rl(RaR) = I_A.$$

By hypothesis, $\sum_{a\in I} rl(RaR)$ is a right annihilator ideal. Since I_A is the smallest right annihilator ideal containing I, we must have the equality †.

Since R is a right SA-ring, the right hand side of the equality ‡ is a right annihilator ideal contained in I^A. To prove the reverse inclusion, let K be a right annihilator ideal contained in I and $x \in K$. Then we have $rl(RxR) \subseteq K \subseteq I$. Thus, $K = \sum_{a\in K} rl(RaR) \subseteq \sum_{rl(RaR)\subseteq I} rl(RaR)$. This shows that $I^A \subseteq \sum_{rl(RaR))\subseteq I} rl(RaR))$. This completes the proof of the equality ‡. □

Lemma 5.11. Let $I_1, I_2, ..., I_n$ be ideals of R such that for each $1 \leq i \neq j \leq n$, I_i and I_j be co-prime. Then for each $1 \leq j \leq n$, I_j and $\bigcap_{i=1,i\neq j}^n I_i$ are co-prime.

Proof. Fix $1 \leq j \leq n$, we have $I_j + I_i = R$ for each $1 \leq i \leq n$ with $i \neq j$. Thus for each $1 \leq i \leq n$ with $i \neq j$, there exist elements $a_{ij} \in I_j$ and $b_i \in I_i$ such that $1 = a_{ij} + b_i$. Multiplying these expressions for all $i \neq j$, we obtain $1 = \prod_{i=1,i\neq j}^n (a_{ij} + b_i)$, which is of the form $x + y$, where $x \in I_j$ and $y \in \bigcap_{i=1,i\neq j}^n I_i$. So we are done. □

Proposition 5.12. Let $I_1, I_2, ..., I_n$ be ideals of a semiprimme ring R such that for each $1 \leq i \neq j \leq n$, I_i and I_j be co-prime. Then $\bigcap_{i=1}^n I_i$ is a right annihilator ideal if and only if each I_j $(1 \leq j \leq n)$ is a right annihilator ideal.

Proof. The necessity is obvious. Conversely, assume that $\bigcap_{i=1}^n I_i$ is a right annihilator ideal. Consider the ideal I_j for some $1 \leq j \leq n$ with $l(I_j) \subseteq l(a)$ and $a \in R$. By Lemma 5.11, I_j and $\bigcap_{k=1,k\neq j}^n I_k$ are co-prime. Therefore $I_j + \bigcap_{k=1,k\neq j}^n I_k = R$. Hence

$1 = x + y$ for some $x \in I_j$ and $y \in \bigcap_{k=1,k\neq j}^{n} I_k$. Multiplying both sides by a, we get $a = ax + ay$. By Lemma 3.5, $l(I_j y) \subseteq l(ay)$. Now, since $I_j y \subseteq \bigcap_{k=1}^{n} I_k$ and $\bigcap_{k=1}^{n} I_k$ is a right annihilator ideal, we have $ay \in \bigcap_{k=1}^{n} I_k$. Thus $ay \in I_j$. Since $ax \in I_j$, we conclude that $a = ax + ay \in I_j$. □

Proposition 5.12 does not hold for an infinite number of co-prime ideals. Consider the ring $\mathbb{Z}$ (i.e., the ring of integer numbers). Then, we have the intersection of all its maximal ideals is zero. But none of them is an annihilator ideal, by Corollary 5.9.

Proposition 5.13. The following statements hold.

(1) Let I be an ideal of a semiprime ring R and P be a prime ideal of R with $I \cap P$ a right annihilator ideal. Then either I or P is a right annihilator ideal.
(2) Let P and Q be prime ideals of a semiprime ring R which do not belong to a chain and $P \cap Q$ is a right annihilator ideal. Then both P and Q are right annihilator ideals.
(3) Let I be an ideal of a semiprime ring R and M be a maximal ideal of R such that $I \not\subseteq M$ and $I \cap M$ is a right annihilator ideal. Then both ideals I and M are right annihilator ideals.

Proof. (1) If $I \subseteq P$, then $I = I \cap P$ is a right annihilator ideal. Now suppose that $I \not\subseteq P$ and $a \in R$ and $l(P) \subseteq l(a)$. Then, there exists $x \in I \setminus P$. By Lemma 3.5, $l(Px) \subseteq l(aRx)$. Since $I \cap P$ is a right annihilator ideal and $Px \subseteq I \cap P$, we have $aRx \subseteq I \cap P$, and hence $aRx \subseteq P$. As $x \notin P$, we have $a \in P$. Hence, P is a right annihilator ideal.

(2) Let $x \in Q \setminus P$ and $l(P) \subseteq l(a)$ for some $a \in R$. Then, $l(Px) \subseteq l(aRx)$, by Lemma 3.5. Since $P \cap Q$ is a right annihilator ideal and $Px \subseteq P \cap Q$, we have $aRx \in P \cap Q$. Hence, $aRx \in P$. This implies $a \in P$. Similarly, we can prove that Q is a right annihilator ideal.

(3) Since $I \not\subseteq M$, I and M are co-prime and hence by Proposition 5.12, I and M are right annihilator ideals. □

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