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OPTIMAL DESIGNS UNDER PROPORTIONAL INTERFERENCE MODELS WHEN THE SIGN OF PROPORTIONALITY CONSTANTS IS KNOWN

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Abstract. Interference models are widely used in block experiments, where a treatment applied to one plot also affects its neighbors. This paper examines two proportional interference models, where the neighbor effects are proportional to the direct treatment effects. These models have the advantage of requiring fewer parameters to estimate but face challenges due to the nonlinearity of the estimates. Although conceptually simple, their technical implementation is somewhat more complex.

In this paper, we examine two-sided and one-sided proportional interference models and transform their nonlinear nature into a more manageable form by assuming that the proportionality constants are known. Our findings provide a closed-form analytical solution for optimal designs under proportional interference models.

1. Introduction

In block experiments, plots are often located close to one another, which can cause interference between treatments applied to adjacent plots. Such neighbor effects arise in many fields, including serology, agriculture, forestry, horticulture, and medicine. In

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general, when experimental units are arranged in close proximity, the response observed for one unit may be influenced by treatments applied to neighboring units.

For example, in varietal crop trials, the yields of shorter varieties may be reduced by shading from taller neighboring varieties [16]. In plants with extensive root systems, early-germinating varieties may absorb nutrients from adjacent plots. Similarly, in pesticide or fungicide experiments, treatment drift may occur downwind, affecting nearby plots [15]. These neighbor influences can result in biased outcomes and decreased efficiency, motivating the development of interference models designed to adjust for such effects.

The concept of neighbor-balanced designs was first introduced by Rees [25], who proposed arranging antigens on a circular plate in serological experiments so that each antigen appeared equally often beside each of the other antigens, thereby minimizing the effects of neighboring plots. Subsequent research has expanded this concept to circular and non-circular layouts, direct or indirect neighboring, and higher-order neighbor structures. Various methods for constructing such designs have been proposed by Lawless [23], Hwang [13], Azais et al. [2], Iqbal et al. [14], and Ahmed and Akhtar [1]. ' In the theory of experimental designs, determining optimal designs is often a primary objective. Various criteria have been used to evaluate the optimality of experimental designs. Kiefer [20] introduced the concept of universally optimal designs, which are optimal according to all optimality criteria (such as A, D, E, etc.).

A substantial body of literature has examined the optimality of circular and non-circular designs under various forms of interference, including one-sided or two-sided neighbor effects, equal or unequal neighbor effects, and correlated or uncorrelated observations. Notable contributions include Druilhet [6], Kunert et al. [22], Filipiak and Markiewicz [9, 10, 11, 12], Fakhari-Esferizi and Pooladsaz [7, 8], and Khodsiani and Fakhari-Esferizi [18].

A proportional interference model was introduced by Li et al. [24], in which the neighbor effects of a treatment are proportional to its direct effect. This model is attractive because it reduces the number of parameters to be estimated, although it introduces nonlinear estimation difficulties. Li et al. [24] investigated two proportional interference models in the context of approximate design theory, deriving optimal measures using Kiefer's equivalence theorems for multiple optimality criteria and highlighting the increasing complexity of symmetric block structures as the block size grows.

Zhang and Kong [27] investigated direct and total effects under proportional interference models for both guarded (border) and unguarded (non-border) plots. Zhang and Kong [27] proposed optimal noncircular designs for guarded proportional-interference models, providing greater flexibility and improved efficiency compared with traditional circular designs. They also developed a parallel framework for unguarded models, thereby filling an important gap in the literature on noncircular proportional-interference designs.

Related work in crossover experiments includes studies by Bailey and Kunert [3] and Zheng [28]. Kempton et al. [17] extended the proportional interference model to cases with an unknown proportionality constant, using linear approximation methods to compute information matrices and identify optimal designs for various values of the constant. Bose and Stufken [4] examined crossover designs assuming a known constant, demonstrating that optimal designs are robust over a broad range of plausible proportionality values and that estimating the constant typically results in only minor efficiency losses.

In this paper, we study two proportional interference models under the assumption that the proportionality constants are known. This assumption avoids the nonlinear estimation difficulties that arise when the constants are unknown and allows for a clearer derivation of the corresponding information matrices. Using the resulting matrix structures, we establish the universal optimality of certain circular equineighbored designs.

Much of the existing literature on proportional interference models has focused on identifying optimal measures within the framework of approximate design theory. However, such optimal proportions are often impractical to implement directly as exact designs. In addition, as the block size increases, the number of distinct symmetric blocks grows rapidly, making the search for optimal proportions computationally demanding. Therefore, determining the theoretical structure of optimal designs becomes essential for simplifying the construction of optimal designs. From a design-theoretic perspective, the ultimate objective is to obtain efficient or optimal exact designs for any number of blocks. The results presented in this paper contribute toward this objective by identifying structural conditions that lead to universally optimal designs. Importantly, the optimal designs depend not on the specific magnitudes of the proportionality constants but only on their signs.

This paper is organized as follows. Section 2 introduces notations and key definitions, including proportional interference models with known proportionality constants. In Section 3, by the obtained structures of information matrices for these models, universal optimality of some circular equineighbored designs is proved, and several examples are provided.

2. Notations and definitions

A design is called circular if each block of the design has a circular form, i.e., there are both left- and right-neighbors for each treatment. In a block design setup, if each block consists of a single line of plots and the blocks are well-separated, border plots are needed to capture the effects of left and right neighbors at the beginning and end of each block. Each border plot receives a treatment that is the same at the opposite end of the block, but it is not used for measuring the response variable (see Druilhet [6]).

Let $\mathcal{D}_{t,b,k}$ denote the class of circular block designs with t treatments, b blocks of size k, with treatment labels $0, 1, \ldots, t-1$. Also, $d(u, j)$ denotes the treatment assigned to plot j of block u for the design d in $\mathcal{D}_{t,b,k}$. By this notation, $d(u, j)$ is said to be the left-neighbor of $d(u, j+\gamma)$ at distance γ, $\gamma = 1, 2, \cdots, k-1$, where $j+\gamma$ is calculated (modulo k)+1. Also, $d(u, j+\gamma)$ is the right-neighbor of $d(u, j)$ at distance γ. It is worth mentioning that if $d(u, j)$ coincides with $d(u, j-\gamma)$ or $d(u, j+\gamma)$ in the circular design d, then it is called a self-neighboring at distance γ, $\gamma = 1, 2, \cdots, k-1$, for that treatment.

Let $\mathbf{T}_{d,u}$, $\mathbf{L}_{d,u} = \mathbf{H}_k\mathbf{T}_{d,u}$, and $\mathbf{R}_{d,u} = \mathbf{H}_k^{'}\mathbf{T}_{d,u}$ be the design matrices of treatment, left- and right-neighbor effects in block u, $u = 1, \ldots, b$, respectively, where $\mathbf{H}_k = (h_{ij})$ is the $k \times k$ circulant matrix with elements,

$$h_{ij} = \begin{cases} 1 & \text{if } (i-j=1) \text{ or } (i=1, j=k) \\ 0 & \text{otherwise.} \end{cases}$$

Also, $\mathbf{T}_d = (\mathbf{T}_{d,1}^{'} : \cdots : \mathbf{T}_{d,b}^{'})^{'}$ is the design matrix of treatment effects. It follows that design matrices of left- and right-neighbor effects are $\mathbf{L}_d = (\mathbf{I}_b \otimes \mathbf{H}_k)\mathbf{T}_d$ and $\mathbf{R}_d = (\mathbf{I}_b \otimes \mathbf{H}_k^{'})\mathbf{T}_d$ where $\mathbf{I}_b$ is the identity matrix of order b and $\otimes$ is the Kronecker product.

We consider the following proportional interference model,

$$\mathbf{y} = (\mathbf{T}_d + \lambda_1\mathbf{L}_d + \lambda_2\mathbf{R}_d)\tau + \mathbf{B}\beta + \varepsilon \tag{2.1}$$

where $\mathbf{y} = (y_{1,1}, \ldots, y_{1,k}, y_{2,1}, \ldots, y_{2,k}, \ldots, y_{b,1}, \ldots, y_{b,k})'$ is the observation vector, τ is t-vector of treatment effects, β is the b-vector of block effects, $\mathbf{B} = \mathbf{I}_b \otimes \mathbf{1}_k$ is the design matrix of block effects, ε is the bk-vector of uncorrelated random errors with $\mathrm{E}(\varepsilon) = \mathbf{0}_{bk}$, $\mathbf{1}_k$ is the k-vector of ones and $\mathbf{0}_{bk}$ is the bk-vector of zeros. Also, known constants λ_1 and λ_2 are proportions of the direct treatment effects where $\lambda_1, \lambda_2 \in [-1, 1]$.

Li et al. [24] derived most of their results under the assumption that the proportionality constants λ_1 and λ_2 lie in the interval $[0, 1]$. However, as noted in their discussion, these constants may in practice take negative values. Li et al. [24] further demonstrated that, under the proportional interference model, the optimal measures obtained via their computational search for $\lambda_1, \lambda_2 \in [-1, 0)$ can differ from those identified for $\lambda_1, \lambda_2 \in [0, 1]$.

In this paper, we are interested in the characterization of universally optimal designs for two cases $\lambda_1, \lambda_2 \in [-1, 0)$ and $\lambda_1, \lambda_2 \in [0, 1]$.

If $\lambda_2 = 0$ in model (2.1), then we have the proportional interference model with left-neighbor effects as below,

$$\mathbf{y} = (\mathbf{T}_d + \lambda_1 \mathbf{L}_d)\tau + \mathbf{B}\beta + \varepsilon. \tag{2.2}$$

Let $r_{d,i}$ be the replication of treatment i in design d. If for a square matrix $\mathbf{A}$, $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$, then by definition of $\mathbf{H}_k$, we define $\mathbf{r}_d^\delta = \mathrm{diag}(r_{d,0}, r_{d,1}, \ldots, r_{d,t-1})$, $\mathbf{S}_d$ and $\mathbf{U}_d$ as below,

$$\mathbf{T}_d^{'} \left(\mathbf{I}_b \otimes \mathbf{H}_k\right)^i \mathbf{T}_d = \begin{cases} \mathbf{r}_d^\delta & \text{if } i = 0 \\ \mathbf{S}_d & \text{if } i = 1 \\ \mathbf{U}_d & \text{if } i = 2. \end{cases} \tag{2.3}$$

The (i, j)-th element of $\mathbf{S}_d$ and $\mathbf{U}_d$ indicates the number of times treatment i is the left-neighbor of treatment j at distances 1 and 2 in d, respectively. Also, the diagonal elements of $\mathbf{S}_d$ and $\mathbf{U}_d$ represent the number of self-neighborings at distances 1 and 2, respectively.

A block design d is called a generalized binary design (GBD) which is introduced by Das and Dey [5], if replication of each treatment in each block is $[\frac{k}{t}]$ or $[\frac{k}{t}] + 1$ where $[m]$ denotes the largest integer not exceeding m. Here we denote $\mathcal{B}_{t,b,k}$ as the class of circular GBDs with t treatments, b blocks of size k.

If for a $d \in \mathcal{B}_{t,b,k}$, (i) $r_{d,i} = r$; $i = 0, 1, \ldots, t-1$, and (ii) every pair of distinct treatments occurs in the same number of blocks (say θ) then d is called a balanced block design (BBD) (see Kiefer [21]).

Throughout this paper, we use some properties of circular equineighbored designs (CEDs) which were introduced by Khodsiani and Pooladsaz [19].

Definition 2.1. A block design is said to be a circular equineighbored design (CED) at level δ , and denoted by CED$(t, b, k; \delta)$, $\delta \leq k-1$, if it is a circular BBD in $\mathcal{D}_{t,b,k}$ that every pair of unordered treatments appears as the δ-th neighbors at the same number of blocks.

Definition 2.2. A circular block design is said to be a circular equineighbored design, CED(t, b, k), if it is CED$(t, b, k; \delta)$ for all $\delta = 1, 2, ..., k-1$.

In a CED(t, b, k), neighborhoods at distance δ are balanced for all $\delta = 1, 2, ..., k-1$, and the directions of the neighborhoods are not important in these designs. We use the notation CED$(t, b, k; 1, 2)$ for a CED$(t, b, k; \delta)$ with $\delta = 1, 2$.

3. Main results

In this paper, the information matrix of design d for estimation of treatment effects is denoted by $\mathbf{C}_d$. Kiefer [20] proved that a design $d^* \in \mathcal{D}_{t,b,k}$ is universally optimal if (i) $\mathbf{C}_{d^*}$ is completely symmetric, i.e., all diagonal elements of $\mathbf{C}_{d^*}$ are equal and all its off-diagonal elements are equal, and (ii) $\mathbf{C}_{d^*}$ has maximal trace over $\mathcal{D}_{t,b,k}$.

We analyze the construction of $\mathbf{C}_d$ under both proportional interference models (2.1) and (2.2) to obtain universally optimal designs.

Suppose that $\mathbf{Q}_B = \mathbf{I}_{bk} - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-}\mathbf{B}' = \mathbf{I}_b \otimes (\mathbf{I}_k - \frac{1}{k}\mathbf{J}_k)$ is the orthogonal projector onto the orthocomplement of the column span of $\mathbf{B}$, where $(\mathbf{B}'\mathbf{B})^{-}$ is a generalized inverse of $\mathbf{B}'\mathbf{B}$ and $\mathbf{J}_k$ is the matrix of ones with order k.

By the least-squares method, it can be shown that under model (2.1), the information matrix for estimating τ is

$$\mathbf{C}_d = (\mathbf{T}_d + \lambda_1 \mathbf{L}_d + \lambda_2 \mathbf{R}_d)' \mathbf{Q}_B (\mathbf{T}_d + \lambda_1 \mathbf{L}_d + \lambda_2 \mathbf{R}_d). \tag{3.1}$$

For the orthogonal matrix $\mathbf{H}_k$, it is clear that $\mathbf{H}_k \mathbf{J}_k = \mathbf{H}_k' \mathbf{J}_k = \mathbf{J}_k$ and $\mathbf{H}_k \mathbf{H}_k' = \mathbf{H}_k' \mathbf{H}_k = \mathbf{I}_k$. From (2.3) and the definitions of $\mathbf{L}_d$, $\mathbf{R}_d$, and $\mathbf{Q}_B$, the information matrix of d in (3.1)

has the following form

$$\mathbf{C}_d = (1+\lambda_1^2+\lambda_2^2)\mathbf{r}_d^{\delta}+(\lambda_1+\lambda_2)(\mathbf{S}_d+\mathbf{S}_d')+\lambda_1\lambda_2(\mathbf{U}_d+\mathbf{U}_d')-\frac{(1+\lambda_1+\lambda_2)^2}{k}\mathbf{N}_d\mathbf{N}_d' \tag{3.2}$$

where $\mathbf{N}_d = \mathbf{T}_d'\mathbf{B}$.

Let $\overline{\mathcal{D}}_{t,b,k}$ denote the subclass of $\mathcal{D}_{t,b,k}$ with no self-neighbors at distance 2. In fact, for every design d in $\overline{\mathcal{D}}_{t,b,k}$, $\mathrm{tr}(\mathbf{U}_d) = 0$.

For positive values of λ_1 and λ_2, identifying the universally optimal design over $\mathcal{D}_{t,b,k}$, if it exists, is more complicated and requires computational and algorithmic search methods. To make the problem tractable, we restrict the search to the subclass of designs with no self-neighbors at distance 2. This restriction allows us to explore the design space efficiently and obtain practically useful results.

Theorem 3.1. *Let d^* be a CED$(t,b,k;1,2)$. It is universally optimal under Model (2.1)*

i. *over the class $\mathcal{D}_{t,b,k}$ when $k \le t$ and $\lambda_1, \lambda_2 \in [-1,1]$,*
ii. *over the class $\mathcal{D}_{t,b,k}$ if the self-neighboring at distance 2 for every treatment of d^* is maximum when $k > t$ and $\lambda_1, \lambda_2 \in [-1,0)$,*
iii. *over the class $\overline{\mathcal{D}}_{t,b,k}$ if the self-neighboring at distance 1 for every treatment of d^* is maximum when $k > t$ and $\lambda_1, \lambda_2 \in [0,1]$.*

Proof. According to (3.2), if the matrices $\mathbf{r}_d^{\delta}, \mathbf{S}_d+\mathbf{S}_d', \mathbf{U}_d+\mathbf{U}_d'$, and $\mathbf{N}_d\mathbf{N}_d$ are completely symmetric, then $\mathbf{C}_d$ is completely symmetric. Therefore, each CED$(t,b,k;1,2)$ has a completely symmetric information matrix.

If c_d is defined as the trace of $\mathbf{C}_d$ and n_d as the trace of matrix $\mathbf{N}_d\mathbf{N}_d'$ then by (3.2), we have,

$$c_d = (1+\lambda_1^2+\lambda_2^2)bk + 2(\lambda_1+\lambda_2)\mathrm{tr}(\mathbf{S}_d) + 2\lambda_1\lambda_2\mathrm{tr}(\mathbf{U}_d) - \frac{(1+\lambda_1+\lambda_2)^2}{k}n_d$$

and thus

$$c_d \le (1+\lambda_1^2+\lambda_2^2)bk + 2(\lambda_1+\lambda_2)\mathrm{tr}(\mathbf{S}_d) + 2\lambda_1\lambda_2\mathrm{tr}(\mathbf{U}_d) - \frac{(1+\lambda_1+\lambda_2)^2}{k}\min_{d\in\mathcal{D}_{t,b,k}} n_d.$$

Shah and Sinha [26] showed that n_d is minimal for BBDs. Therefore, the universally optimal design is a design in $\mathcal{B}_{t,b,k}$. Khodsiani and Pooladsaz [19] showed that for every design d in $\mathcal{B}_{t,b,k}$,

$$n_d = bt((t-s)h^2 + s(h+1)^2) \tag{3.3}$$

where $k = ht + s$ such that h and s are two non-negative integers and $s < t$. If $k \leq t$, there is no self-neighboring for the treatments in every design d in $\mathcal{B}_{t,b,k}$ and by (3.3),

$$\max_{d\in\mathcal{D}_{t,b,k}} c_d = (1+\lambda_1^2+\lambda_2^2)bk - \frac{(1+\lambda_1+\lambda_2)^2}{k}bts.$$

Therefore d^* is universally optimal over $\mathcal{D}_{t,b,k}$. But if $k > t$ then there exists some self-neighboring in every design d in $\mathcal{B}_{t,b,k}$, and we consider two different situations separately, as follows:

Situation 1. If $\lambda_1, \lambda_2 \in [-1, 0)$, then $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$. Therefore, the trace of $\mathbf{C}_d$ will be maximal over $\mathcal{D}_{t,b,k}$ when the self-neighboring of each treatment of d at distances 1 and 2 is minimal and maximal, respectively.

Situation 2. If $\lambda_1, \lambda_2 \in [0, 1]$, then $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are both nonnegative. Since $\mathrm{tr}(\mathbf{U}_d) = 0$ for every $d \in \overline{\mathcal{D}}_{t,b,k}$, the trace of $\mathbf{C}_d$ will be maximal over $\overline{\mathcal{D}}_{t,b,k}$ when self-neighboring of each treatment of d at distances 1 is maximal. □

By the results of Khodsiani and Pooladsaz [19], a CED(t, b, k) with $k \leq t$ satisfies the following properties: (i) $\frac{bk}{t}$ is an integer, (ii) each neighboring matrix at distance δ, $\delta = 1, 2, \cdots, \lfloor\frac{k}{2}\rfloor$, is in the form $a(\mathbf{J}_t - \mathbf{I}_t)$ where $a = \dfrac{2bk}{t(t-1)}$.

By this point, the number of blocks required to construct a CED at levels 1 and 2 is equal to the number of blocks of constructing a CED(t, b, t), and therefore, it is recommended to use Khodsiani and Pooladsaz [19] construction methods of CEDs to build the optimal design d^* in Theorem 3.1.

Eample 3.2. Consider an experiment with $t = 4$ fertilizer types to be tested on fields with different slopes (the blocks), where each sloped field can accommodate 8 fertilizer types ($k = 8$). In practical situations, a highly effective fertilizer may stimulate vigorous plant growth, causing a plant to compete more intensely with its neighbors for sunlight, water, and soil nutrients, thereby inducing negative interference effects. By Khodsiani and Pooladsaz [19], we need 3 blocks to construct a CED with t=4 and k=8. The following design is a CED$(4, 3, 8; 1, 2)$:

$$d_1^* = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 3 & 2 & 3 \\ 0 & 2 & 0 & 2 & 3 & 1 & 3 & 1 \\ 0 & 3 & 0 & 3 & 1 & 2 & 1 & 2 \end{pmatrix},$$

with each row representing a block and it satisfies:

$$\mathbf{S}_{d_1^*} + \mathbf{S}_{d_1^*}^{'} = 4(\mathbf{J}_5 - \mathbf{I}_5), \qquad \mathbf{U}_{d_1^*} + \mathbf{U}_{d_1^*}^{'} = 2\mathbf{J}_5 + 4\mathbf{I}_5.$$

The design d_1^* is universally optimal over $\mathcal{D}_{4,3,8}$ for model (2.1) with negative proportionality constants.

Eample 3.3. The following design is a BBD such that $\mathbf{S}_{d_2^*} + \mathbf{S}_{d_2^*}^{'} = \mathbf{U}_{d_2^*} + \mathbf{U}_{d_2^*}^{'} = \mathbf{J}_5 - \mathbf{I}_5$,

$$d_2^* = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3. \end{pmatrix}$$

The design d_2^* is a CED$(5, 2, 5; 1, 2)$ and it is universally optimal over $\overline{\mathcal{D}}_{5,2,5}$ for model (2.1) with $\lambda_1, \lambda_2 \in [0, 1]$.

To obtain the information matrix of the proportional one-sided interference model (2.2), λ_2 is replaced by zero in (3.2) we obtain

$$\mathbf{C}_d = (1 + \lambda_1^2)\mathbf{r}_d{}^{\delta} + \lambda_1(\mathbf{S}_d + \mathbf{S}_d{}^{'}) - \frac{(1 + \lambda_1)^2}{k}\mathbf{N}_d\mathbf{N}_d' \tag{3.4}$$

and thus,

$$\begin{aligned} c_d &= (1 + \lambda_1^2)bk + 2\lambda_1 \mathrm{tr}(\mathbf{S}_d) - \frac{(1 + \lambda_1)^2}{k} n_d \\ &\leq (1 + \lambda_1^2)bk + 2\lambda_1 \mathrm{tr}(\mathbf{S}_d) - \frac{(1 + \lambda_1)^2}{k} \min_{d \in \mathcal{D}_{t,b,k}} n_d. \end{aligned}$$

In fact, under model (2.2), the information matrix of a design depends only on the neighboring matrix at distance 1. Therefore, the self-neighboring at distance 1 of the treatments in a design d is important for evaluating c_d and the following corollary can be proved.

Corollary 3.4. *Let d^* be a CED$(t, b, k; 1)$. Under model (2.2), d^* is universally optimal over $\mathcal{D}_{t,b,k}$*

i. *if $k \leq t$ when $\lambda_1 \in [-1, 1]$,*
ii. *if there is no self-neighboring at distance 1 when $k > t$ and $\lambda_1 \in [-1, 0)$,*
iii. *if the self-neighboring at distance 1 for every treatment of d^* is maximum when $k > t$ and $\lambda_1 \in [0, 1)$.*

Eample 3.5. Following the methodology of Khodsiani and Pooladsaz [19], we construct the following design

$$d_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 0 & 0 \\ 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 0 & 0 & 1 & 1 \\ 3 & 3 & 3 & 4 & 4 & 4 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 4 & 4 & 4 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 3 & 3 \\ 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 0 & 2 & 2 & 4 & 4 \\ 2 & 2 & 2 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 & 0 & 2 & 2 & 2 & 4 & 4 & 1 & 1 \\ 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 2 & 2 \end{pmatrix}$$

such that $\mathbf{S}_{d_3^*} + \mathbf{S}'_{d_3^*} = 22\mathbf{I}_5 + 10\mathbf{J}_5$ and by Corollary 3.4, it is universally optimal in $\mathcal{D}_{5,10,13}$ under model (2.2) with positive λ_1.

Conclusion

In a block experiment, when the treatment assigned to a plot also influences its neighboring plots and these influences are proportional to its direct effect, the proportional interference models are noticed. The proportional interference models have the advantage of requiring fewer parameters to estimate, but face challenges due to the nonlinearity of the estimates.

In this paper, in order to obtain closed-form analytical solutions for optimal designs, we assume that the constants λ_1 and λ_2 in the proportional interference model are known. We consider the proportional interference model with left- and right-neighbor effects, where the neighbor effects are proportional to the direct treatment effects. For this model, we show that the direction of neighboring is not important, while neighbor balance up to distance 2 plays a crucial role. For the cases of positive and negative λ_1 and λ_2, separately, we prove that certain CEDs at levels 1 and 2 are universally optimal.

Furthermore, for the proportional interference model with only left-neighbor effects, our results show that balance in neighborings is required only at distance 1. For positive λ_1, a CED$(t, b, k; 1)$ with maximum self-neighboring at distance 1, and for negative λ_1, a CED$(t, b, k; 1)$ with no self-neighboring at distance 1 are universally optimal designs.

Much of the existing literature on proportional interference models has focused on identifying optimal measures within the framework of approximate design theory. However, such optimal proportions are often impractical to implement directly as exact designs. In addition, as the block size increases, the number of distinct symmetric blocks grows rapidly, making the search for optimal proportions computationally demanding. Therefore, determining the theoretical structure of optimal designs becomes essential for simplifying the construction of optimal designs. The main key outcome of our results is that our optimal designs are not obtained approximately, and they don't depend on the exact numerical values of the proportionality constants. In fact, only the signs of λ_1 and λ_2 affects the structure of the optimal design and in practice, the sign of these constants is typically known. Also, since there is no restriction on the number of t and k in our results and the necessary number of blocks for constructing our optimal designs is not too large, these designs are available and applicable.

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