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ON SOLUTION OF CONFORMABLE FRACTIONAL INTEGRAL EQUATION VIA MEASURE OF NONCOMPACTNESS AND FIXED POINT APPROACH

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With hearts heavy with sorrow, we dedicate this article to the 168 students under the age of twelve from Shajareh Tayyebeh School in Minab–innocent children who lost their lives in Iran. Their only "offense" was being present in a classroom, striving to learn, to grow, and to dream-an unbearable truth that challenges the conscience of our modern world.

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Abstract. This study seeks to explore the solutions of a class of non-linear equations via fractional conformable integral in the defined Banach space of continuous functions. In this setting, employing the technique of noncompactness measures, we utilize Petryshyn's fixed point theorem to establish the existence of solutions for this particular fractional equation under specific weaker conditions related to the space. Additionally, an example is provided to demonstrate the applicability of the results obtained from our theorem.

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1. Introduction

The examination of functional equations has attracted considerable interest because of theirs extensive applicability in multiple scientific domains, such as physics, engineering, and biological systems [1–5]. Fractional integrals and derivatives serve as valuable instruments for simulating various processes and phenomena, finding numerous applications across diverse fields such as science, bioengineering, and mathematics. Numerous researchers have investigated different forms of fractional integral equations ($\mathbb{FIE}$s) [6–10].

Given the significance of fixed point theory, numerous researchers have utilized it to establish the existence and uniqueness of solutions for $\mathbb{IE}$s and differential equations, resulting in a considerable number of publications in this area. Moreover, the presence of solutions for certain functional $\mathbb{IE}$s involving Riemann-Liouville (RL) type fractional integral, which emerge in various applied issues, has been examined in [11–15].

In this work, we specifically focus on the existence theorem for solutions to generalized conformable $\mathbb{FIE}$s as follows for $\tau \in \mathsf{E}_{\mathrm{b}} := [\mathrm{b}, \mathrm{b}_1]$,

$$\mathfrak{p}(\tau) = \chi\left(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \left[\frac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \frac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\right), \tag{1.1}$$

in Banach space $(\mathcal{E}, \|\cdot\|)$, $\mathcal{E} = C(\mathsf{E}_{\mathrm{b}})$, $\mathrm{b} = 0$, with the uniform norm $\|\wp\| = \sup\big\{|\wp(z)| : z \in \mathsf{E}_{\mathrm{b}}\big\}$, where $\mathfrak{p} \in \mathcal{E}$, $\chi \in C\big(\mathsf{E}_{\mathrm{b}} \times \mathbb{R}^3\big)$, $k \in C\big(\mathsf{E}_{\mathrm{b}}^2 \times \mathbb{R}\big)$ and α, β, $\theta \in C(\mathsf{E}_{\mathrm{b}}, \mathsf{E}_{\mathrm{b}})$. In fact, the left-fractional conformable integral operator of order υ, $\Re(\upsilon) > 0$ and parameter $\gamma > 0$, is described by

$$\mathscr{I}^{\upsilon;\gamma}\mathfrak{p}(\tau) = \int_{\mathrm{b}}^{\tau} \left[\frac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \frac{\mathfrak{p}(s)}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s, \qquad \mathrm{b} < \tau < \mathrm{b}_1,$$

with the famous Gamma function $\Gamma(\alpha)$ [16, 17]. The conformable $\mathbb{FI}$ operator generalizes the RL $\mathbb{FI}$s ([1, p. 44]), Hadamard $\mathbb{FI}$s ([18]) and Katugampola $\mathbb{FI}$s [19]. The basic tools used in this investigation is Petryshyn's fixed point theorem (FPT) as a generalization of FPT of Darbo in which are based on the technique of measure of noncompactness (MNC) [20, 21]. It seems that, the idea of using FPT of Petryshyn to solve nonlinear functional $\mathbb{IE}$s was introduced by Kazemi *et al.* for first time [22]. Recently many authors have applied this FPT to obtain existence results for nonlinear functional $\mathbb{IE}$s in Banach algebra [23–26]. The advantage of Petryshyn's FPT 2.3 among the others (Darbo and Schauder FPTs) lies in the fact that in applying the theorem, one does not need to verify

the involved operator maps a closed convex subset onto itself. Also, the third condition is the bounded condition (Theorem 3.1-(III)) which shows that the 'sublinear condition' discussed in several works in the literature holds. The strengths of our research work can be outlined as follows: when we consider $b = 0$, $\gamma = 1$, and $\gamma = 0$, the conformable $\mathbb{FIE}$ (1.1) will become to the Katugampola $\mathbb{FI}$s, the the RL $\mathbb{FI}$s and Hadamard $\mathbb{FI}$s, respectively.

The remainder of the paper is structured as follows: In Section 2, we introduce some preliminaries and describe concepts of the tools we need. Section 3 is devoted to stating and proving an existence theorem for $\mathbb{FIE}$ (1.1) involving condensing operators using the FPT of Petryshyn. In Section 4, we offer an example that verify the application of this kind of non-linear functional $\mathbb{IE}$s.

2. Auxiliary facts and results

We review key foundational concepts and auxiliary information related to the theory of noncompactness in a Banach space $(\mathcal{E}, \|\cdot\|)$. The MNC of a bounded subset $\varnothing = \Omega \subseteq \mathcal{E}$ is formulated by ([27]),

$$\Sigma_{\mathsf{MNC}}(\Omega) = \inf\Big\{\varepsilon > 0 \,:\, \Omega \subseteq \overset{m}{\underset{j=1}{\cup}} V_j,\, \dim(V_j) \leq \varepsilon\Big\}.$$

The next theorem shows the essential properties of Kuratowski's MNC.

Theorem 2.1 ([21]). *The* MNC *satisfies the following properties for bounded sets* $\Omega, \grave{\Omega}$ *in* $\mathcal{E}$*:*

i) *For the closure of* Ω, $\Sigma_{\mathsf{MNC}}(\overline{\Omega}) = \Sigma_{\mathsf{MNC}}(\Omega)$*;*
ii) $\Sigma_{\mathsf{MNC}}(\Omega) = 0 \Leftrightarrow \overline{\Omega}$ *is compact;*
iii) $\Omega \subseteq \grave{\Omega}$ *implies* $\Sigma_{\mathsf{MNC}}(\Omega) \leq \Sigma_{\mathsf{MNC}}(\grave{\Omega})$*;*
iv) $\Sigma_{\mathsf{MNC}}(\Omega \cup \grave{\Omega}) = \max\big\{\Sigma_{\mathsf{MNC}}(\Omega), \Sigma_{\mathsf{MNC}}(\grave{\Omega})\big\}$*;*
v) $\Sigma_{\mathsf{MNC}}(\lambda\Omega) = |\lambda|\, \Sigma_{\mathsf{MNC}}(\Omega)$, *where* $\lambda\Omega = \big\{\lambda z \,:\, z \in \Omega, \lambda \in \mathbb{R}\big\}$, $\lambda \in \mathbb{R}$*;*
vi) $\Sigma_{\mathsf{MNC}}(\Omega + \grave{\Omega}) \leq \Sigma_{\mathsf{MNC}}(\Omega) + \Sigma_{\mathsf{MNC}}(\grave{\Omega})$, *where* $\Omega + \grave{\Omega} = \big\{z + \grave{z} \,:\, z \in \Omega,\, \grave{z} \in \grave{\Omega}\big\}$.

Let $\varepsilon > 0$ be given. The modulus of continuity of a bounded subset $\varnothing \neq \Omega \subseteq \mathcal{E}$, for an element $\wp \in \Omega$ is expressed by

$$\varpi(\wp, \varepsilon) = \sup\Big\{\big|\wp(z) - \wp(\grave{z})\big| \,:\, |z - \grave{z}| \leq \varepsilon,\, z, \grave{z} \in \mathsf{E}_{\mathrm{b}}\Big\}.$$

Theorem 2.2 ([28]). *The structure of Kuratowski's* MNC *for* $\varnothing = \Omega \subseteq \mathcal{E}$ *has equivalent description as follows:*

$$\Sigma_{\mathsf{MNC}}(\Omega) = \lim_{\varepsilon \to 0} \sup \Big\{ \varpi(\wp, \varepsilon) \ : \ \wp \in \Omega \Big\}. \tag{2.1}$$

A map $\mathfrak{p} \in C(\mathcal{E}, \mathcal{E})$ is known as a k-set contraction if, for every $\varnothing \neq \Omega \subset \mathcal{E}$, $\mathfrak{p}(\Omega)$ is also bounded and there exists $0 < \varsigma < 1$ such that $\Sigma_{\mathsf{MNC}}\big(\mathfrak{p}(\Omega)\big) \leq \varsigma \Sigma_{\mathsf{MNC}}(\Omega)$ [29]. Further, $\mathfrak{p}$ is said to be densifying or condensing map when for each $\Sigma_{\mathsf{MNC}}(\Omega) > 0$, the inequality $\Sigma_{\mathsf{MNC}}\big(\mathfrak{p}(\Omega)\big) < \Sigma_{\mathsf{MNC}}(\Omega)$ is fulfilled [29]. Note that, a k-set contraction with $0 < \varsigma < 1$, is densifying, but the converse is not true. A densifying mapping $\mathfrak{p} : \bar{B}_\delta \to \mathcal{E}$ satisfies the boundary condition, if $\mathfrak{p}(z) = \ell z$ for some $z \in \partial B_\delta = \big\{ \wp \in \mathcal{E} \ : \ \|\wp\| = \delta \big\}$, $\ell \leq 1$ holds [30].

Theorem 2.3 ([21]). *The* FP*s' set of a densifying mapping* $\mathfrak{p} : \bar{B}_r \to \mathcal{E}$ *is nonempty.*

3. Main results

The goal of our results is to examine the existence of solutions to the nonlinear functional $\mathbb{FIE}$ (1.1) by using Theorem 2.3 as key tool.

Theorem 3.1. *Suppose that*

i) *There exist* $\hat{c}_1, \hat{c}_2, \hat{c}_3 \geq 0$ *with* $\hat{c}_1 + \hat{c}_2 < 1$ *such that*

$$\big|\chi(\tau, z_1, z_2, z_3) - \chi(\tau, \bar{z}_1, \bar{z}_2, \bar{z}_3)\big| \leq \sum_{j=1}^{3} \hat{c}_j \big|z_j - \bar{z}_j\big|;$$

ii) χ *satisfies the bounded condition for* $r_0 \geq 0$ *as follows:*

$$\sup \Big\{ \big|\chi(\tau, z_1, z_2, z_3)\big| \ : \ \tau \in \mathsf{E}_{\mathrm{b}}, -\delta_0 \leq z_1, z_2 \leq \delta_0, |z_3| \leq \tfrac{\widehat{k}\mathrm{b}_1^{\upsilon\gamma}}{\gamma^\upsilon \Gamma(1+\upsilon)} \Big\} \leq \delta_0,$$

with, $\widehat{k} = \sup \big\{ \big|k(\tau, \dot{\tau}, z)\big| \ : \ \forall\, \tau, \dot{\tau} \in \mathsf{E}_{\mathrm{b}},\ z \in [-\delta_0, \delta_0] \big\}$.

Then, $\mathbb{FIE}$ (1.1) *has at least one solution in* $\mathcal{E}$.

Proof. First, we create the operator $\Phi : B_{\delta_0} \to \mathcal{E}$ by

$$(\Phi\mathfrak{p})(\tau) = \chi\Big(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^\gamma - (s-\mathrm{b})^\gamma}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau, s, \eth(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\, \mathrm{d}s\Big),$$

and show that Φ is continuous by considering a few steps.

Step 1. Consider $\mathfrak{p}, \grave{\mathfrak{p}} \in B_{\delta_0}$ with $\|\mathfrak{p} - \grave{\mathfrak{p}}\| \leq \varepsilon$ for a given $\varepsilon > 0$. Then,

$$\begin{aligned}
\big|(\Phi\mathfrak{p})(\tau) - (\Phi\grave{\mathfrak{p}})(\tau)\big| &= \bigg|\chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \\
&\quad \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&\quad - \chi\bigg(\tau, \grave{\mathfrak{p}}\big(\alpha(\tau)\big), \grave{\mathfrak{p}}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\grave{\mathfrak{p}}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\bigg| \\
&\leq \bigg|\chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&\quad - \chi\bigg(\tau, \grave{\mathfrak{p}}\big(\alpha(\tau)\big), \grave{\mathfrak{p}}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\grave{\mathfrak{p}}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\bigg| \\
&\quad + \bigg|\chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&\quad - \chi\bigg(\tau, \grave{\mathfrak{p}}\big(\alpha(\tau)\big), \grave{\mathfrak{p}}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\grave{\mathfrak{p}}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\bigg| \\
&\quad + \bigg|\chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&\quad - \chi\bigg(\tau, \grave{\mathfrak{p}}\big(\alpha(\tau)\big), \grave{\mathfrak{p}}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\grave{\mathfrak{p}}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\bigg| \\
&\leq \hat{c}_1\big|\mathfrak{p}\big(\alpha(\tau)\big) - \grave{\mathfrak{p}}\big(\alpha(\tau)\big)\big| + \hat{c}_2\big|\mathfrak{p}\big(\beta(\tau)\big) - \grave{\mathfrak{p}}\big(\beta(\tau)\big)\big| \\
&\quad + \hat{c}_3 \int_{\mathrm{b}}^{\tau} \Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{|k(\tau,s,\mathfrak{p}(\theta(s)))-k(\tau,s,\grave{\mathfrak{p}}(\theta(s)))|}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s \\
&\leq (\hat{c}_1 + \hat{c}_2)\big\|\mathfrak{p} - \grave{\mathfrak{p}}\big\| + \tfrac{\hat{c}_3 \mathrm{b}_1^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}\varpi(k,\varepsilon),
\end{aligned}$$

for $\tau \in \mathsf{E}_{\mathrm{b}}$, where

$$\varpi(k,\varepsilon) = \sup\Big\{\big|k(\tau,\grave{\tau},z) - k(\tau,\grave{\tau},v)\big| \,:\, \tau,\grave{s} \in \mathsf{E}_{\mathrm{b}},\ -\delta_0 \leq z, v \leq \delta_0,\ \|z-v\| \leq \varepsilon\Big\}.$$

This implies $\varpi(k,\varepsilon) \to 0$ as $\varepsilon \to 0$, since $k = k(\tau,\grave{\tau},z)$ is uniformly continuous on $\mathsf{E}_{\mathrm{b}}^2 \times [-\delta_0,\delta_0]$ which confirms the continuity of Φ.

Step 2. Now, for $\tau, \grave{\tau} \in \mathsf{E}_{\mathrm{b}}$ with $\grave{\tau} - \tau \leq \varepsilon$, $\varepsilon > 0$, and $\mathfrak{p}$ in a bounded subset $\Omega \subseteq \mathcal{E}$, we have

$$\begin{aligned}
\big|(\Phi\mathfrak{p})(\grave{\tau}) - (\Phi\mathfrak{p})(\tau)\big| &= \bigg|\chi\bigg(\grave{\tau}, \mathfrak{p}\big(\alpha(\grave{\tau})\big), \mathfrak{p}\big(\beta(\grave{\tau})\big), \\
&\quad \int_{\mathrm{b}}^{\grave{\tau}} \Big[\tfrac{(\grave{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\grave{\tau},s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)
\end{aligned}$$

$$
\begin{aligned}
&- \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\Big| \\
\leq &\Big|\chi\bigg(\dot{\tau}, \mathfrak{p}\big(\alpha(\dot{\tau})\big), \mathfrak{p}\big(\beta(\dot{\tau})\big), \int_{\mathrm{b}}^{\dot{\tau}} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\dot{\tau},s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&- \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\Big| \\
&+ \Big|\chi\bigg(\dot{\tau}, \mathfrak{p}\big(\alpha(\dot{\tau})\big), \mathfrak{p}\big(\beta(\dot{\tau})\big), \int_{\mathrm{b}}^{\dot{\tau}} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\dot{\tau},s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&- \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\Big| \\
&+ \Big|\chi\bigg(\dot{\tau}, \mathfrak{p}\big(\alpha(\dot{\tau})\big), \mathfrak{p}\big(\beta(\dot{\tau})\big), \int_{\mathrm{b}}^{\dot{\tau}} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\dot{\tau},s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg) \\
&- \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau} \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\Big| \\
\leq\ &\hat{c}_4\bigg|\int_{\mathrm{b}}^{\dot{\tau}} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\dot{\tau},s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s \\
&- \int_{\mathrm{b}}^{\tau} \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg| \\
&+ \hat{c}_2\big|\mathfrak{p}\big(\beta(\dot{\tau})\big) - \mathfrak{p}\big(\beta(\tau)\big)\big| + \hat{c}_1\big|\mathfrak{p}\big(\alpha(\dot{\tau})\big) - \mathfrak{p}\big(\alpha(\tau)\big)\big| + \varpi(\chi,\varepsilon) \\
\leq\ &\tfrac{\hat{c}_4}{\Gamma(\upsilon)} \int_0^{\tau} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{|k(\dot{\tau},s,\mathfrak{p}(\theta(s)))-k(\tau,s,\mathfrak{p}(\theta(s)))|}{(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s \\
&+ \tfrac{\hat{c}_4}{\Gamma(\upsilon)} \int_0^{\tau} \bigg| \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{(s-\mathrm{b})^{1-\gamma}} \\
&- \left[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{(s-\mathrm{b})^{1-\gamma}}\bigg|\,\mathrm{d}s \\
&+ \tfrac{\hat{c}_4}{\Gamma(\gamma)} \int_{\tau}^{\dot{\tau}} \left[\tfrac{(\dot{\tau}-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\right]^{\upsilon-1} \tfrac{|k(\dot{\tau},s,\mathfrak{p}(\theta(s)))|}{(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s \\
&+ \hat{c}_2\varpi\big(\mathfrak{p}, \varpi(\beta,\varepsilon)\big) + \hat{c}_1\varpi\big(\mathfrak{p}, \varpi(\alpha,\varepsilon)\big) + \varpi(\chi,\varepsilon).
\end{aligned}
$$

where

$$\varpi\left(\chi,\varepsilon\right) = \sup\Big\{\big|\chi(\tau,z_1,z_2,z_3) - \chi(\dot{\tau},z_1,z_2,z_3)\big| \,:\, |\tau-\dot{\tau}| \leq \varepsilon,\ \tau,\dot{\tau} \in \mathsf{E}_{\mathrm{b}},$$

$$z_1, z_2 \in [-\delta_0, \delta_0],\ |z_3| \leq \tfrac{\widehat{k}\mathrm{b}_1^{\gamma\upsilon}}{\gamma^\upsilon\Gamma(1+\upsilon)}\Big\}.$$

Therefore,

$$\begin{aligned}
\big|(\Phi\mathfrak{p})(\grave{\tau}) - (\Phi\mathfrak{p})(\tau)\big| &\leq \tfrac{\hat{c}_4\varpi(k,\varepsilon)}{\gamma^\upsilon\Gamma(1+\upsilon)}\Big[(\grave{\tau}-\mathrm{b})^{\gamma\upsilon} - \big((\grave{\tau}-\mathrm{b})^\gamma - (\tau-\mathrm{b})^\gamma\big)^\upsilon\Big] \\
&\quad + \tfrac{\hat{c}_4\widehat{k}}{\gamma^\upsilon\Gamma(1+\upsilon)}\Big[(\tau-\mathrm{b})^{\gamma\upsilon} - (\grave{\tau}-\mathrm{b})^{\gamma\upsilon} + \big((\grave{\tau}-\mathrm{b})^\gamma - (\tau-\mathrm{b})^\gamma\big)^\upsilon\Big] \\
&\quad + \tfrac{\hat{c}_4\widehat{k}}{\gamma^\upsilon\Gamma(1+\upsilon)}\big((\grave{\tau}-\mathrm{b})^\gamma - (\tau-\mathrm{b})^\gamma\big)^\upsilon \\
&\quad + \hat{c}_2\varpi\big(\mathfrak{p}, \varpi(\beta,\varepsilon)\big) + \hat{c}_1\varpi\big(\mathfrak{p}, \varpi(\alpha,\varepsilon)\big) + \varpi(\chi,\varepsilon) \\
&\leq \tfrac{\hat{c}_4\varpi(k,\varepsilon)}{\gamma^\upsilon\Gamma(1+\upsilon)}(\tau-\mathrm{b})^{\gamma\upsilon} + \tfrac{\hat{c}_4\widehat{k}}{\gamma^\upsilon\Gamma(1+\upsilon)}\big[(\grave{\tau}-\mathrm{b})^\gamma - (\tau-\mathrm{b})^\gamma\big]^\upsilon \\
&\quad + \hat{c}_2\varpi\big(\mathfrak{p}, \varpi(\beta,\varepsilon)\big) + \hat{c}_1\varpi\big(\mathfrak{p}, \varpi(\alpha,\varepsilon)\big) + \varpi(\chi,\varepsilon), \\
&\leq \tfrac{\hat{c}_4\varpi(k,\varepsilon)}{\gamma^\upsilon\Gamma(1+\upsilon)}(\tau-\mathrm{b})^{\gamma\upsilon} + \tfrac{\hat{c}_4\widehat{k}}{\gamma^\upsilon\Gamma(1+\upsilon)}(\varepsilon)^\upsilon \\
&\quad + \hat{c}_2\varpi\big(\mathfrak{p}, \varpi(\beta,\varepsilon)\big) + \hat{c}_1\varpi\big(\mathfrak{p}, \varpi(\alpha,\varepsilon)\big) + \varpi(\chi,\varepsilon).
\end{aligned}$$

Hence, $\varpi(\Phi\mathfrak{p},\varepsilon) \leq (\hat{c}_1+\hat{c}_2)\varpi(\mathfrak{p},\varepsilon)$, for $\mathfrak{p}\in\Omega$. Taking limit as $\varepsilon\to 0$, we get

$$\mu(\Phi\Omega) \leq (\hat{c}_1+\hat{c}_2)\Sigma_{\mathsf{MNC}}(\Omega).$$

Indeed, Φ is a densifying map.

`Step 3.` Let $\mathfrak{p}\in\partial\bar{B}_{\delta_0}$. If $\Phi\mathfrak{p}=\ell\mathfrak{p}$, then $\ell\delta_0 = \ell\|\mathfrak{p}\| = \|\Phi\mathfrak{p}\|$ and taking into account condition (ii), we obtain

$$\big|\Phi\mathfrak{p}(\tau)\big| = \left|\chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_{\mathrm{b}}^{\tau}\Big[\tfrac{(\tau-\mathrm{b})^\gamma-(s-\mathrm{b})^\gamma}{\gamma}\Big]^{\upsilon-1}\tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\bigg)\right| \leq \delta_0,$$

for each $\tau\in\mathsf{E}_\mathrm{b}$, hence $\|\Phi\mathfrak{p}\|\leq\delta_0$, which shows $\ell\leq 1$. The proof is complete. □

Corollary 3.2. *Suppose*

K1) $\alpha,\beta,\theta\in C(\mathsf{E}_\mathrm{b},\mathsf{E}_\mathrm{b})$, $k_j\in C(\mathsf{E}_\mathrm{b}\times\mathbb{R})$ *and there is* $\varsigma_j>0$, *such that*

$$\big|k_i(\tau,z)-k_i(\tau,\grave{z})\big| \leq \varsigma_i|z-\grave{z}|, \qquad z,\grave{z}\in\mathbb{R},\ \tau\in\mathsf{E}_\mathrm{b},$$

and $|k_j(\tau,0)|\leq\aleph_j$, $j=1,2$;

K2) $k(\tau,\grave{\tau},z)\in C(\mathsf{E}_\mathrm{b}^2\times\mathbb{R})$ *and there is* m_1,m_2 *and* p *such that*

$$\big|k(\tau,\grave{\tau},z)\big| \leq m_1+m_2|z|^p, \qquad \tau,\grave{\tau}\in\mathsf{E}_\mathrm{b},\ z\in\mathbb{R}.$$

Then $\mathbb{FIE}$ *of the form,*

$$\mathfrak{p}(\tau)=k_1\big(\tau,\mathfrak{p}(\beta(\tau))\big)+\tfrac{k_2(\tau,\mathfrak{p}(\alpha(\tau)))}{\Gamma(\upsilon)}\int_{\mathrm{b}}^{\tau}\Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1}\tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s,\quad \tau\in\mathsf{E}_{\mathrm{b}},$$

has at least one solution in $\mathcal{E}$*, if*

$$\aleph_1+\tfrac{\mathrm{b}_1^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}(m_1+m_2)(\varsigma_2+\aleph_2)+\varsigma_1<1.$$

Proof. Put $Q(\tau,z_1,z_2,z_3)=h_1(\tau,z_1)+k_2(\tau,z_2)z_3$, where $z_1=\mathfrak{p}\big(\alpha(\tau)\big)$, $z_2=\mathfrak{p}\big(\beta(\tau)\big)$, and

$$z_3=\int_{\mathrm{b}}^{\tau}\Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1}\tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s.$$

Clearly, (K1) confirms the hypothesis (i) holds. Next, we check the correctness of (ii). Let, $\|\mathfrak{p}\|\leq\delta_0$, $\delta_0>0$ and put $M_2=m_1+m_2\delta_0^p$. Then,

$$\begin{aligned}
|\beth(\tau)|&=\sup\Big\{\big|Q(\tau,z_1,z_2,z_3)\big|\ :\ \tau\in\mathsf{E}_{\mathrm{b}},-\delta_0\leq z_1,z_2\leq\delta_0,\\
&\quad -M_2\tfrac{\mathrm{b}_1^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}\leq z_3\leq M_2\tfrac{\mathrm{b}_1^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}\Big\}\\
&=\Big|k_1\big(\tau,\mathfrak{p}(\beta(\tau))\big)+\tfrac{k_2(\tau,\mathfrak{p}(\alpha(\tau)))}{\Gamma(\upsilon)}\int_{\mathrm{b}}^{\tau}\Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1}\tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\Big|\\
&\leq\Big|k_1\big(\tau,\mathfrak{p}(\beta(\tau))\big)-k_1\big(\tau,0\big)\Big|+\big|k_1\big(\tau,0\big)\big|+\big|k_2\big(\tau,\mathfrak{p}(\alpha(\tau))\big)-k_2(\tau,0)\big|\\
&\quad+\big|k_2(\tau,0)\big|\Big|\int_{\mathrm{b}}^{\tau}\Big[\tfrac{(\tau-\mathrm{b})^{\gamma}-(s-\mathrm{b})^{\gamma}}{\gamma}\Big]^{\upsilon-1}\tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(s-\mathrm{b})^{1-\gamma}}\,\mathrm{d}s\Big|\\
&\leq\varsigma_1\|\mathfrak{p}\|+\aleph_1+\tfrac{\tau^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}(\aleph_2+l\|\mathfrak{p}\|)\big[m_1+m_2\|\mathfrak{p}\|^p\big].
\end{aligned}$$

Hence, the real number $\delta_0>0$ in (ii) satisfies

$$\varsigma_1\delta_0+\aleph_1+\tfrac{\tau^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}(\aleph_2+\delta_0)\big[m_1+m_2\delta_0^p\big]\leq\delta_0.$$

We define a function $\psi\in C([0,1])$ as follows:

$$\psi(r_0)=\varsigma_1\delta_0+\aleph_1+\tfrac{\tau^{\gamma\upsilon}}{\gamma^{\upsilon}\Gamma(1+\upsilon)}(\aleph_2+\varsigma_2\delta_0)[m_1+m_2\delta_0^p]-\delta_0.$$

Then $\psi(0)>0$ and $\psi(1)<0$ by (K2) and since ψ is continuous, there is $\delta_0\in(0,1)$ such that $\psi(\delta_0)=0$. This shows that this inequality has a solution in $(0,1)$. The proof is complete. □

The following is the celebrated quadratic $\mathbb{IE}$ of Chandrasekhar type in [31, 32] which is applied in the theories of radiative transfer, neutron transport, and kinetic energy of gases [33, 34].

Corollary 3.3. *Let* $Q(\tau, z_1, z_2, z_3) = 1 + z_1 z_3$, *on* E_b, $\mathrm{b} = 0$, $\gamma = \upsilon = 1$, $\alpha(\tau) = \tau$, $\mathfrak{p}\big(\alpha(\tau)\big) = \mathfrak{p}(\tau)$ *and*

$$k(\tau, \grave{\tau}, z) = \tfrac{\tau}{\tau+\grave{\tau}}\phi(\grave{\tau})z.$$

Then, $\mathbb{FIIE}$ (1.1) *has the following form*

$$\mathfrak{p}(\tau) = 1 + \mathfrak{p}(\tau)\int_0^\tau \tfrac{\tau}{\tau+s}\phi(s)\mathfrak{p}(s)\,\mathrm{d}s, \qquad \tau \in \mathsf{E}_\mathrm{b}.$$

Corollary 3.4. *If* $\mathrm{b} = 0$, $\gamma = 1$, *then* $\mathbb{FIIE}$ (1.1) *can be written*

$$\mathfrak{p}(\tau) = \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_0^\tau \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)(\tau-s)^{1-\upsilon}}\,\mathrm{d}s\bigg), \qquad \tau \in \mathsf{E}_\mathrm{b}.$$

Corollary 3.5. *If* $\mathrm{b} = 0$, *then* $\mathbb{FIIE}$ (1.1) *can be changed to*

$$\mathfrak{p}(\tau) = \chi\bigg(\tau, \mathfrak{p}\big(\alpha(\tau)\big), \mathfrak{p}\big(\beta(\tau)\big), \int_0^\tau \Big[\tfrac{\tau^\gamma - s^\gamma}{\gamma}\Big]^{\upsilon-1} \tfrac{k(\tau,s,\mathfrak{p}(\theta(s)))}{\Gamma(\upsilon)s^{1-\gamma}}\,\mathrm{d}s\bigg), \qquad \tau \in \mathsf{E}_\mathrm{b}.$$

4. Application

We examine our mentioned results with an example of functional $\mathbb{FIIE}$.

Example 1. Consider the following nonlinear $\mathbb{FIIE}$:

$$\begin{aligned}\mathfrak{p}(\tau) &= \tfrac{1}{10}\tau e^{-(\tau+3)} + \tfrac{\tau\mathfrak{p}(\tau^3)}{6(1+\tau)} + \tfrac{|\mathfrak{p}(1-s)|}{\sqrt{7}+|\mathfrak{p}(1-s)|}\\ &\int_0^\tau \Big[\tfrac{\tau^\gamma - s^\gamma}{\gamma}\Big]^{\upsilon-1} \tfrac{e^{-3\tau^2}\big(se^\tau + \frac{1}{4}\ln\big(1+\big|\mathfrak{p}(\sqrt{\tau})\big|\big)\big)}{7\Gamma(\upsilon)s^{1-\gamma}}\,\mathrm{d}s, \quad \tau \in \mathsf{E}_\mathrm{b},\ \mathrm{b} = 0,\ \mathrm{b}_1 = 1,\end{aligned} \tag{4.1}$$

for two cases as follows:

$$\begin{aligned}&\mathtt{Case\,1.} \qquad \upsilon \in \left\{\tfrac{3}{5}, \tfrac{5}{6}, \tfrac{10}{11}\right\}, \quad \gamma = \tfrac{1}{4},\\ &\mathtt{Case\,2.} \qquad \gamma \in \left\{\tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}\right\}, \quad \upsilon = \tfrac{1}{3}.\end{aligned}$$

The given problem (4.1) is a special case of $\mathbb{FIIE}$ (1.1). Let us take $\chi : \mathsf{E}_\mathrm{b} \times \mathbb{R}^3 \to \mathbb{R}$, $\alpha, \beta, \theta : \mathsf{E}_\mathrm{b} \to \mathsf{E}_\mathrm{b}$ and $k : \mathsf{E}_\mathrm{b}^2 \times \mathbb{R} \to \mathbb{R}$ and comparing (4.1) with $\mathbb{FIIE}$ (1.1), we get $\alpha(\tau) = \tau^3$, $\beta(\tau) = 1 - \tau$, $\theta(\tau) = \sqrt{\tau}$, and

$$\widehat{k} = \sup\Big\{\Big|e^{-3\tau^2}\left(se^\tau + \tfrac{1}{4}\ln(1+|\mathfrak{p}|)\right)\Big| : \forall\,\tau \in \mathsf{E}_\mathrm{b},\ \mathfrak{p} \in [-\delta_0, \delta_0]\Big\}.$$

Clearly, these functions satisfy hypothesis (i), since,

$$\begin{aligned}\big|\chi(\tau, z_1, z_2, z_3) - \chi(\tau, \bar{z}_1, \bar{z}_2, \bar{z}_3)\big| &= \Big|\tfrac{1}{10}\tau e^{-(\tau+3)} + \tfrac{\tau z_1}{6(1+\tau)} + \tfrac{|z_2|}{\sqrt{7}+|z_2|} + \tfrac{z_3}{7}\\ &\quad - \Big[\tfrac{1}{10}\tau e^{-(\tau+3)} + \tfrac{\tau\bar{z}_1}{6(1+\tau)} + \tfrac{|\bar{z}_2|}{\sqrt{7}+|\bar{z}_2|} + \tfrac{\bar{z}_3}{7}\Big]\Big|\end{aligned}$$

$$\leq \left|\tfrac{\tau z_1}{6(1+\tau)} - \tfrac{\tau \bar{z}_1}{6(1+\tau)}\right| + \left|\tfrac{|z_2|}{\sqrt{7}+|z_2|} - \tfrac{|\bar{z}_2|}{\sqrt{7}+|\bar{z}_2|}\right| + \left|\tfrac{z_3}{7} - \tfrac{\bar{z}_3}{7}\right|$$
$$\leq \tfrac{1}{6}\left|z_1 - \bar{z}_1\right| + \tfrac{1}{\sqrt{7}}\left|z_2 - \bar{z}_2\right| + \tfrac{1}{7}\left|z_j - \bar{z}_j\right|$$

with $\hat{c}_1 = \frac{1}{6}$, $\hat{c}_2 = \frac{1}{\sqrt{7}}$, $\hat{c}_3 = \frac{1}{7}$ and $\hat{c}_1 + \hat{c}_2 < 1$. We check the next hypothesis (ii). For $\|\mathfrak{p}\| \leq \delta_0$, we have

$$|\mathfrak{p}(\tau)| = \left|\tfrac{1}{10}\tau e^{-(\tau+3)} + \tfrac{\tau \mathfrak{p}(\tau^3)}{6(1+\tau)} + \tfrac{|\mathfrak{p}(1-s)|}{\sqrt{7}+|\mathfrak{p}(1-s)|}\right.$$
$$\left. + \int_0^\tau \left[\tfrac{\tau^\gamma - s^\gamma}{\gamma}\right]^{\upsilon-1} \tfrac{e^{-3\tau^2}\left[se^{\tau} + \frac{1}{4}\ln\left(1+\left|\mathfrak{p}\left(\sqrt{\tau}\right)\right|\right)\right]}{7\Gamma(\upsilon)s^{1-\gamma}}\, \mathrm{d}s\right| \leq \delta_0,$$

for $\tau \in \mathsf{E}_\mathrm{b}$. Next, for different predicted cases, we review the challenges in order to confirm the existence of a solution.

`Case 1`. From the given data in this case of $\mathbb{FIE}$ (4.1), hypothesis (ii) holds, if

$$\mathcal{A}_1 = \tfrac{1}{10}e^{-3} + \tfrac{1}{6}\delta_0 + \tfrac{1}{\sqrt{7}}\delta_0 + \tfrac{1}{7\Gamma(1+\upsilon)\gamma^\upsilon}\left(1 + \tfrac{1}{4}\delta_0\right) \simeq \left\{\begin{array}{ll} 0.2172, & \upsilon = 3/5, \\ 0.2814, & \upsilon = 5/6, \\ 0.3400, & \upsilon = 10/11, \end{array}\right\} \leq \delta_0,$$

with the following estimated $\delta_0 \geq 0.4, 0.5, 0.6$ for $\upsilon = \frac{3}{5}$, $\frac{5}{6}$, $\upsilon = \frac{10}{11}$, respectively. We plot the variation of $\mathcal{A}_1$ in Fig. 1 when $\upsilon \in \left\{\frac{3}{5}, \frac{5}{6}, \frac{10}{11}\right\}$, respectively. In fact, as the order of the derivative υ approaches 1, the value of $\mathcal{A}_1$ increases. These estimated results are shown in Table 1. Hence (ii) holds. Therefore, thanks to Theorem 3.1, $\mathbb{FIE}$ (4.1) has at least

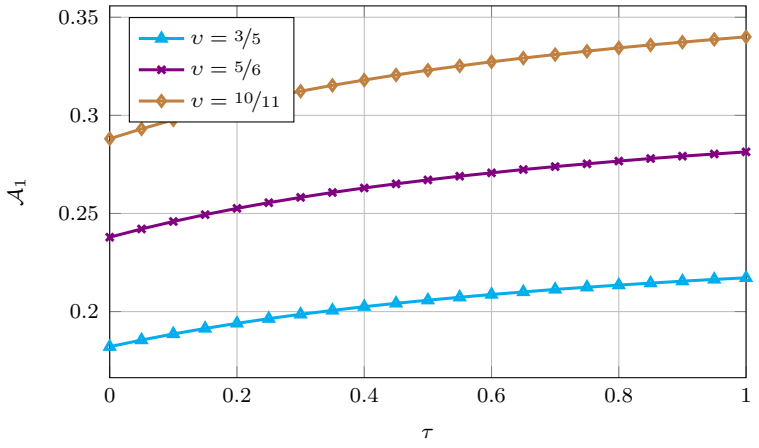


FIGURE 1. 2D-plots of $\mathcal{A}_1$ for $\mathbb{FIE}$ (4.1) with three values of derivative order υ in `Case 1`.

TABLE 1. Estimated results of $\mathcal{A}$ for $\mathbb{FIE}$ (4.1) for three different values of derivative order υ in `Case 1`.

τ	$\mathcal{A}_1$		
	$\upsilon = 3/5$	$\upsilon = 5/6$	$\upsilon = 10/11$
0.00	0.1821	0.2379	0.2881
0.05	0.1855	0.2421	0.2931
0.10	0.1886	0.2459	0.2977
0.15	0.1914	0.2494	0.3018
0.20	0.1940	0.2526	0.3056
0.25	0.1964	0.2555	0.3091
0.30	0.1986	0.2582	0.3123
0.35	0.2006	0.2607	0.3153
0.40	0.2025	0.2630	0.3180
0.45	0.2042	0.2651	0.3206
0.50	0.2058	0.2671	0.3230
0.55	0.2073	0.2690	0.3252
0.60	0.2087	0.2707	0.3273
0.65	0.2100	0.2724	0.3292
0.70	0.2113	0.2739	0.3310
0.75	0.2124	0.2753	0.3327
0.80	0.2135	0.2767	0.3344
0.85	0.2145	0.2780	0.3359
0.90	0.2155	0.2792	0.3373
0.95	0.2164	0.2803	0.3387
1.00	0.2172	0.2814	0.3400

one solution in $\mathcal{E}$ for `Case 1`.

`Case 2`. From the given data in this case of $\mathbb{FIE}$ (4.1), hypothesis (ii) holds, if

$$\mathcal{A}_2 = \tfrac{1}{10}e^{-3} + \tfrac{1}{6}\delta_0 + \tfrac{1}{\sqrt{7}}\delta_0 + \tfrac{1}{7\Gamma(1+\upsilon)\gamma^{\upsilon}}\left(1 + \tfrac{1}{4}\delta_0\right) \simeq \left\{ \begin{array}{ll} 0.1334, & \upsilon = 3/5, \\ 0.1838, & \upsilon = 5/6, \\ 0.2330, & \upsilon = 10/11, \end{array} \right\} \leq \delta_0,$$

with the following estimated $\delta_0 \geq 0.25, 0.35, 0.45$, for $\upsilon = \frac{3}{5}, \frac{5}{6}, \frac{10}{11}$, respectively. We plot the variation of $\mathcal{A}_2$ in Fig. 2 when $\gamma \in \left\{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}\right\}$, respectively. In fact, as the parameter of the derivative γ approaches 0, the value of $\mathcal{A}_2$ decreases. One can see these estimated results in Table 2. Hence (ii) holds. Therefore, thanks to Theorem 3.1 $\mathbb{FIE}$ (4.1) has at least one solution in $\mathcal{E}$ for `Case 2`.

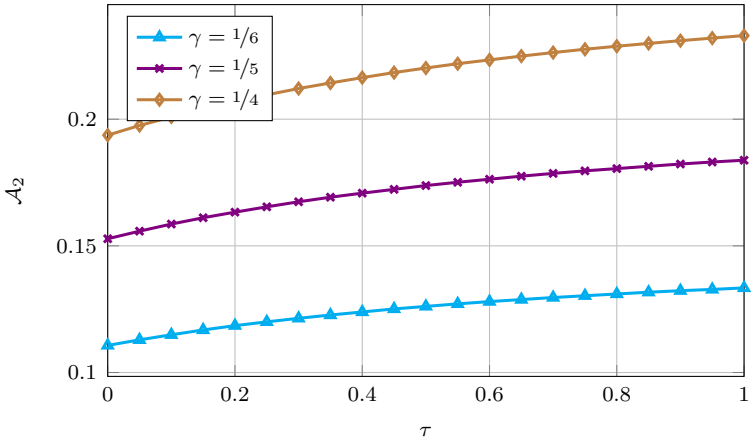


FIGURE 2. 2D-plots of $\mathcal{A}_2$ for $\mathbb{FIE}$ (4.1) with three values of derivative parameter γ in `Case 2`.

TABLE 2. Estimated results of $\mathcal{A}_2$ for $\mathbb{FIE}$ (4.1) for three different values of derivative parameter γ in `Case 2`.

τ	$\mathcal{A}_2$		
	$\boldsymbol{\gamma = 1/6}$	$\boldsymbol{\gamma = 1/5}$	$\boldsymbol{\gamma = 1/4}$
0.00	0.1107	0.1528	0.1937
0.05	0.1129	0.1558	0.1975
0.10	0.1149	0.1586	0.2009
0.15	0.1168	0.1611	0.2041
0.20	0.1185	0.1633	0.2070
0.25	0.1200	0.1654	0.2096
0.30	0.1214	0.1674	0.2121
0.35	0.1227	0.1692	0.2143
0.40	0.1239	0.1708	0.2164
0.45	0.1251	0.1723	0.2184
0.50	0.1261	0.1738	0.2202
0.55	0.1271	0.1751	0.2219
0.60	0.1280	0.1763	0.2234
0.65	0.1288	0.1775	0.2249
0.70	0.1296	0.1786	0.2263
0.75	0.1303	0.1796	0.2276
0.80	0.1310	0.1805	0.2288
0.85	0.1317	0.1814	0.2299
0.90	0.1323	0.1823	0.2310
0.95	0.1328	0.1831	0.2320
1.00	0.1334	0.1838	0.2330

5. Conclusion

In this paper, we have thoroughly investigated the solvability of solutions for a class of nonlinear $\mathbb{FIE}$s. By utilizing the MNC in conjunction with Petryshyn's FPT, we established the existence of solutions within the space of bounded continuous functions $\mathcal{E}$. Our findings highlight the importance of condensing operators in resolving complex functional $\mathbb{FIE}$s. The results obtained in this study contribute significantly to the ongoing development of techniques for analyzing complex functional $\mathbb{FIE}$s. Future research may focus on further generalizing these methods to higher-order and more intricate forms of hybrid functional equations, with potential applications in domains such as biological modeling, traffic systems, and control theory.

Abbreviations

FPT: Fixed-point theorems
$\mathbb{FIE}$: Fractional integral equation
$\mathbb{IE}$: Integral equation
MNC: Measure of non-compactness

Availability of Data and Materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

MK: Methodology, validation, formal analysis, actualization, investigation, and preparation of the initial draft. **HRS**: Validation, formal analysis, review and edit manuscript. **MES**: Actualization, methodology, formal analysis, validation, investigation, software, simulation, review, edit manuscript and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

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