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INVERSE GAMMA-F FAMILY OF DISTRIBUTIONS

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Abstract. In this paper, a new general family of continuous distributions, named the inverse gamma-F family of distributions, is introduced. Some properties of the new family of distributions are derived. The parameters are estimated by maximum likelihood method and the information matrix is obtained. A member of the family, the inverse gamma-Weibull distribution is defined and studied. Some properties of this distribution such as hazard function, quantile function, Shannon entropy, moments, the distribution of the order statistics, maximum likelihood estimation and the observed information matrix are studied. Also, a application based on real data set is presented to illustrate the proposed distribution.

1. Introduction

Statistical distributions have been extensively used over the past decade in describing, predicting and modeling data in different areas. Great amount of effort has been focus on introducing of generalized distributions or the new family of distributions which are more flexible to modeling data. A recent technique for generating a family of meaningful distributions is the generator approach. Let X_F be a random variable with the probability density function (pdf) $f(x)$ and the cumulative density function (cdf) $F(x)$ and T be a

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continuous random variable with pdf $r(t)$. The generator approach defines a new family of distributions with cdf as

$$G(x) = \int_{-\infty}^{W(F(x))} r(t)\ dt \tag{1.1}$$

where $W(F(x))$ is positive, continuse and finite conditions (see Alzaatreh et al [1]). The cdf $G(x)$ in (1.1) can be written as $G(x) = R\{W(F(x))\}$, where $R(t)$ is the cdf of the random variable T. It is clear that the form of obtained distribution in (1.1) depend on the choice of the generator distribution, $r(t)$, and cdf F. Some cases of this family are: the beta-G family of Eugene and Famoye [4] and Jonse [6], the gamma-G family of Zografos and Balakrishnan [10] and the kum-G family of Cordeiro and Castro [2].

This paper is to aims to introduce a new family of univariate continuous distributions through the generator distribution of the inverse-gammah (IG) distribution. The contents are organized as follows. In Section 2, we define the inverse gamma-G family of distributions and derive some of it's general mathematical properties. Additionally, the likelihood estimation and the information matrix are presented. In Section 3 we discuss some members of inverse gamma-F (IGF) family. The inverse gamma-Weibull (IGW) distribution in detail is discussed in Section 4. In this Section the properties of inverse gamma-Weibull distribution have been obtained in more detail; including hazard function, quantile function, Shannon entropy, moments, order statistics, maximum likelihood estimation and the observed information matrix. A application based on a real data set is carried out in Section 5. Finally, the paper ends with concluding remark in Section 6.

2. The Inverse Gamma-G Family

Suppose that the generator random variable T follows the IG distribution with the parameters α and β, $r(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, t^{-(\alpha+1)}\, e^{\frac{-\beta}{t}}\ \ t > 0$, with cdf $R(t) = \frac{\Gamma(\alpha, \beta t^{-1})}{\Gamma(\alpha)}$ where $\Gamma(a, b)$ is the upper incomplete gamma function given by $\Gamma(a, b) = \int_0^b t^{a-1} e^{-t} dt$. From (1.1), the cdf of IGF family is defined as

$$G(x; \boldsymbol{\theta}) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{-\log(1-F(x))} x^{-(\alpha+1)}\, e^{\frac{-\beta}{t}}\, dt = R\{-\log(1 - F(x))\}. \tag{2.1}$$

where $\boldsymbol{\theta} = (\alpha, \beta)$. The family pdf reduces to

$$g(x; \theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, f(x)\ \frac{-\log(1 - F(x)))^{-(\alpha+1)}}{1 - F(x)}\ \exp\Big\{\frac{\beta}{\log(1 - F(x))}\Big\}, \quad x \in R. \tag{2.2}$$

We call each member of the new family of distributions generated from (2.2) as "inverse gamma-G distribution". In the remainder of this Section, we will focus on some statistical properties of the inverse gamma-F family of distributions.

(a) The pdf in (2.2) can be written as $g(x;\boldsymbol{\theta}) = h(x)\, r(H(x))$, where $h(x)$ and $H(x)$ are hazard and cumulative hazard functions of the random variable X_F with cdf $F(x)$. Hence, this family of distributions can be considered as a family of distributions arising from a weighted hazard functions.

(b) The relationship between the random variables X_G with pdf (2.2) and the generator random variable T is given by $X_G = F^{-1}(1 - e^{-T})$. This provides an easy way for simulating random variable X_G by first simulating random variable T from pdf $r(t)$ and computing $X_G = F^{-1}(1-e^{-T})$, which has the cdf $G(x;\boldsymbol{\theta})$. Also, $E_{X_G}[X^k]$ can be obtained as

$$\begin{aligned} E_{X_G}[X^k] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{-\infty}^{\infty} x^k\, f(x)\ \frac{-\log(1-F(x)))^{-(\alpha+1)}}{1-F(x)}\ \exp\left\{\frac{\beta}{\log(1-F(x))}\right\} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} [F^{-1}(1-e^{-t})\,]^k\ t^{-(\alpha+1)}\ e^{\frac{-\beta}{t}}\ dt \\ &= E_T[F^{-1}(1-e^{-T})\,]^k\,. \end{aligned}$$

(c) According to the cdf in (2.1), the quantile function, $Q(\lambda)$, $0 < \lambda < 1$, of the IGF family of distributions can be computed by using the formula

$$Q(\lambda) = F^{-1}(1 - e^{-R^{-1}(\lambda)})\,.$$

(d) The entropy of a random variable is a measure of variation of uncertainty. Entropy has been used in various situations in science and engineering. The Shannon entropy of

the IGG family of distributions is obtained as

$$\begin{aligned}\eta_{X_G} &= E[-\log(g(x;\theta)] = -E\Big[\alpha\log(\beta) - \log(\Gamma(\alpha)) - (\alpha+1)\log(-\log(1-F(X)))\\ &+ \log(\frac{f(X)}{1-F(X)}) + \frac{\beta}{\log(1-F(X))}\Big]\\ &= -\alpha\log(\beta) + \log(\Gamma(\alpha)) - \int_{-\infty}^{\infty}\Big[-(\alpha+1)\log(-\log(1-F(x))) + \log(\frac{f(x)}{1-F(x)})\\ &+ \frac{\beta}{\log(1-F(x))}\Big]\, g(x;\theta)\, dx\\ &= -\alpha\log(\beta) + \log(\Gamma(\alpha)) - \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \Big[t - \frac{\beta}{t} - (\alpha+1)\log(t)\\ &+ \log(f(F^{-1}(1-e^{-t})))\Big] t^{-(\alpha+1)}\ e^{\frac{-\beta}{t}}\ dt\\ &= -E[\log f(F^{-1}(1-e^{-T}))] - \frac{\beta}{\alpha-1} + \log(\beta\Gamma(\alpha)) + \alpha - (\alpha+1)\psi(\alpha)\end{aligned}$$

where $\psi(\alpha) = \frac{\partial\log(\Gamma(\alpha))}{\partial\alpha}$ is the digamma function. By using the mean $\mu_T = \frac{\beta}{\alpha-1}$ and the Shannon entropy $\eta_T = \log(\beta\Gamma(\alpha)) + \alpha - (\alpha+1)\psi(\alpha)$ of the generator random variable T with the inverse gamma distribution, then

$$\eta_{X_G} = -E\big[\log f(F^{-1}(1-e^{-T}))\big] - \mu_T + \eta_T. \tag{2.3}$$

2.1. **Maximum Likelihood Estimation.** Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the new family of distributions from complete samples. Let x_1, x_2, ...,x_n be observed values from the IGG distribution with density (2.2), then the the log-likelihood function can be written as

$$\begin{aligned}\ell(\theta) &= n\alpha\log(\beta) - n\log(\Gamma(\alpha)) - (\alpha+1)\sum_{i=1}^n \log\Big(-\log(1-F(x_i))\Big)\\ &+ \beta\sum_{i=1}^n\Big(\log(1-F(x_i))\Big)^{-1} + \sum_{i=1}^n\log\Big(\frac{f(x_i)}{1-F(x_i)}\Big).\end{aligned}$$

Solving equations

$$\frac{\partial\ell}{\partial\alpha} = n\log(\beta) - n\psi(\alpha) - \sum_{i=1}^n\log(-\log(1-F(x_i))) = 0$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n\alpha}{\beta} + \sum_{i=1}^n(\log(1-F(x_i)))^{-1} = 0.$$

We obtain the MLEs of α , β by the following recursive procedure:

$$\widehat{\alpha_{j+1}} = \frac{-\widehat{\beta}_j}{n}\sum_{i=1}^{n}(\log\left(1-F(x_i)\right))^{-1} \text{ and } \widehat{\beta}_{j+1} = \exp\left\{\psi(\widehat{\alpha}_j)+\frac{1}{n}\sum_{i=1}^{n}\log(-\log(1-F(x_i)))\right\}$$

with some proper initial points $\widehat{\alpha}_0$ and $\widehat{\beta}_0$ The information matrix, whose inverse is the asymptotic variance-covariance matrix of MLEs $\widehat{\alpha}$ and $\widehat{\beta}$, is obtained as a 2×2 matrix $I(\boldsymbol{\theta}) = (I_{ij})$, $i = 1, 2$, $j = 1, 2$, $\boldsymbol{\theta} = (\alpha, \beta)^T$, where

$$\begin{aligned}
I_{11} &= -E\Big[\frac{\partial^2}{\partial\alpha^2}\log g(X;\theta)\Big] = \psi'(\alpha)\\
I_{22} &= -E\Big[\frac{\partial^2}{\partial\beta^2}\log g(X;\theta)\Big] = \frac{\alpha}{\beta^2}\\
I_{12} = I_{21} &= -E\Big[\Big(\frac{\partial}{\partial\alpha}\log g(X;\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\beta}\log g(X;\boldsymbol{\theta})\Big)\Big]\\
&= E\Big[-\frac{\alpha}{\beta}\left(\log(\beta)-\psi(\alpha)\right)-\frac{\log(-\log(1-F(X)))}{\log(1-F(X))}\Big]\\
&= -\frac{\alpha}{\beta}\left(\log(\beta)-\psi(\alpha)\right)+\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\log(t)\;t^{-(\alpha+2)}\;e^{\frac{-\beta}{t}}\;dt\\
&= -\frac{\alpha}{\beta}\left(\log(\beta)-\psi(\alpha)\right)+\frac{\beta^{\alpha+)}}{\Gamma(\alpha)}\int_0^{\infty}\log(\beta u)\;(\beta u)^{-(\alpha+2)}\;e^{\frac{-1}{u}}\;du\\
&= -\frac{\alpha}{\beta}\left(\log(\beta)-\psi(\alpha)\right)+\frac{\alpha}{\beta}\left(\log(\beta)-\psi(\alpha+1)\right)\\
&= \frac{\alpha}{\beta}\left(\psi(\alpha)-\psi(\alpha+1)\right)
\end{aligned}$$

3. Some Members of Inverse Gamma-G Family

In this Section, we look at to two distributions which are defined as the special cases of generated distributions in the IGF family.

3.1. **Inverse Gamma-Exponential Distribution.**

The cdf of the inverse gamma-exponential distribution when $f(x) = \eta\, e^{-\eta x}$ in equation (2.2) is defined as

$$G(x;\boldsymbol{\theta}) = \frac{\Gamma(\alpha,\beta\,(\eta x)^{-1})}{\Gamma(\alpha)}\,,\quad \alpha,\beta,\eta>0\,,\quad x>0\,.$$

The corresponding pdf is

$$g(x;\boldsymbol{\theta}) = \frac{\beta\,\eta^{-1}}{\Gamma(\alpha)}\;x^{-(\alpha+1)}\;\;e^{\frac{-\beta}{\eta x}}$$

It is clear that the new generated distribution is inverse gamma distribution with parameters α and $\frac{\beta}{\theta}$. The generator distribution is form-invariant, in the sense that if the

generator random variable of T follows $r(t;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a scaler or vectors of parameters, then there exists a function $\tau = \tau(\boldsymbol{\theta})$, such that

$$g(x;\boldsymbol{\theta}) = r(x;\tau).$$

3.2. **Inverse Gamma-Extreme Value Distribution.** Consider the extreme value distribution with pdf and cdf given by $f(x) = \frac{1}{\sigma}\, e^{\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}}$, $x \in R$ and $F(x) = 1 - e^{\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}}}$. Then, the inverse gamma-extreme value (IGEV) distribution with parameters $\boldsymbol{\theta} = (\alpha, \beta, \mu, \sigma)^T$ has the cdf given by

$$G(x;\boldsymbol{\theta}) = \frac{\Gamma(\alpha, \beta\, e^{-\frac{x-\mu}{\sigma}})}{\Gamma(\alpha)}\,, \quad \alpha, \beta, \sigma > 0\,,\ \mu \in R\,, \quad x \in R$$

and the corresponding pdf becomes

$$g(x;\boldsymbol{\theta}) = \frac{\beta^\alpha}{\sigma\,\Gamma(\alpha)}\, e^{-\alpha(\frac{x-\mu}{\sigma}) - \beta e^{-\frac{x-\mu}{\sigma}}}\,.$$

The IGEV distribution includes the generalized Gumbel distribution (Gumbel [5]) when $\alpha = \beta$.

4. Inverse Gamma-Weibull Distribution

In this Section, we defined another of the generated distributions in the IGG family. The inverse gamma-Weibull density is obtained from (2.2) by taking $F(x)$ and $f(x)$ to be the cdf and pdf of the Weibull distribution with k shape parameter and η scale parameter. The cdf of the inverse gamma-Weibull distribution with four parameters $\boldsymbol{\theta} = (\alpha, \beta, k, \eta)^T$, abbreviated as (IGW) distribution, is

$$G(x;\boldsymbol{\theta}) = \frac{\Gamma(\alpha, \beta(\frac{\eta}{x})^k)}{\Gamma(\alpha)}\,, \quad \alpha, \beta, k, \eta > 0\,, \quad x > 0\,. \tag{4.1}$$

The probability density function of the IGW distribution is

$$g(x;\boldsymbol{\theta}) = \frac{k\,(\beta\,\eta^k)^\alpha}{\Gamma(\alpha)}\; x^{-(k\alpha+1)}\; e^{-\beta(\frac{\eta}{x})^k}\,. \tag{4.2}$$

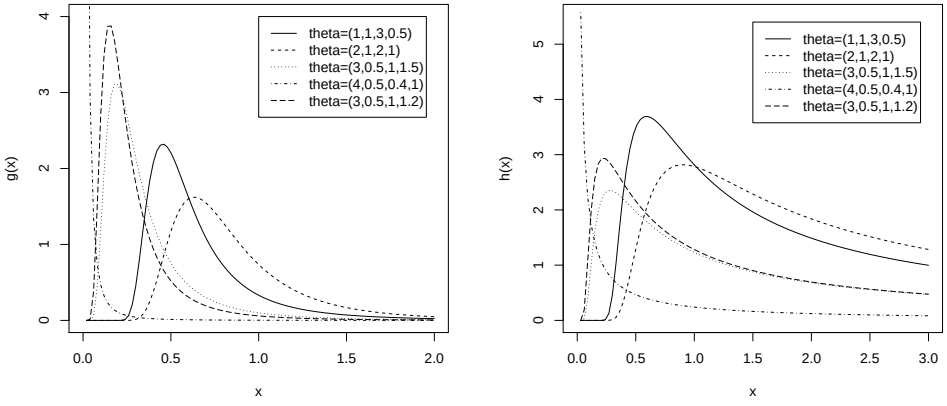
with the hazard function

$$h(x) = \frac{k\,(\beta\,\eta^k)^\alpha}{\Gamma(\alpha) - \Gamma(\alpha, \beta(\frac{\eta}{x})^k)}\; x^{-(k\alpha+1)}\; e^{-\beta(\frac{\eta}{x})^k}\,.$$

Figure 1 shows the graph of the hazard function of IGW distribution for various values of α, β, k, η. This Figure illustrate that hazard function can take different shape such as unimodal and right-skewed.

The IGW distribution has two scale (β, η) and two shape (α, k) parameters and generalizes several well known distributions. The following is a list of well known sub-models of the IGW distribution. The IGW distribution includes the generalized inverse Weibull (GIW) distribution given by de Gusmao et al [3] for $\alpha = 1$. When $\beta = 1$, the IGW distribution reduces to the inverse of generalized gamma (IGG) distribution given by Stacy [9]. Also, for $\beta = 1$, this model includes many distributions such as inverse gamma (IG) if $k = 1$, inverse Weibull (IW) if $\alpha = 1$, inverse Rayleigh (IR) if $\alpha = 1$, $k = 2$, inverse exponential (IE) if $\alpha = k = 1$, inverse Maxwell (IM) if $\alpha = \frac{3}{2}$, $k = 2$, inverse chi-square (ICS) if $\alpha = \frac{n}{2}$, $k = 1$, $\eta = 2$.

One of the advantages of the IGW distribution is that it has a closed form cdf, which can be used to generate random numbers from it by using the simple formula $X = \eta\, T^{\frac{1}{k}}$, where T is the generator random variable with the inverse gamma distribution.



(A) Density function (B) Hazard function

FIGURE 1. The density function and the hazard function the IGW density function for some parameter values

4.1. **Some Properties of IGW Distribution. Quantile function:** Due to the cdf of the IGW distribution in (4.1), the quantile function is the solution of the equation $\Gamma(\alpha, \beta(\frac{\eta}{Q(p)})^k) = p\,\Gamma(\alpha)$.

Shannon entropy: By corresponding to the pdf (f) and cdf (F) of the Weibull distribution, calculating and substituting

$$E[\log(F^{-1}\,(1 - e^{-T}))\,] = \log(\frac{k}{n}) - (1 - \frac{1}{k})\,(\psi(\alpha) - \log(\beta)) - \frac{\beta}{\alpha - 1}$$

in (4), the Shannon entropy of the IGW distribution is given by

$$E[-\log(X)] = \alpha - \log(\frac{k\beta^{\alpha}}{\eta\,\Gamma(\alpha)}) - \frac{k\alpha+1}{k}\left(\psi(\alpha) - \log(\beta)\right) .$$

Order Statistics: Assume $X_1, \ldots, X_n$ are a random sample of size n from the IGW distribution. The pdf of the ith order statistic $X_{i:n}$, $g_{i:n}(x;\boldsymbol{\theta})$, is

$$\begin{aligned} g_{i:n}(x;\boldsymbol{\theta}) &= \frac{1}{B(i,n-i+1)}\, g(x)\, [\, G(x;\boldsymbol{\theta})\,]^{i-1}\, [1-G(x;\boldsymbol{\theta})\,]^{n-i} \\ &= \frac{k(\beta\eta^k)^{\alpha}}{B(i,n-i+1)\,(\Gamma(\alpha))^n}\Big[\Gamma(\alpha,\beta(\frac{\eta}{x})^k))\Big]^{i-1}\Big[\Gamma(\alpha)-\Gamma(\alpha,\beta(\frac{\eta}{x})^k)\Big]^{n-i} \\ &\times x^{-(k\alpha+1)}\, e^{-\beta(\frac{\eta}{x})^k} . \end{aligned}$$

4.2. **Moments.** The rth moment of the IGW distribution is obtained as

$$E[X^r] = \frac{k\,(\beta\,\eta^k)^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{-(k\alpha-r+1)}\; e^{-(\frac{\eta}{x})^k}\; dx \tag{4.3}$$

$$= \frac{(\eta\,\beta^{\frac{1}{k}})^r\;\Gamma(\alpha-\frac{r}{k})}{\Gamma(\alpha)}, \quad \alpha > \frac{r}{k}. \tag{4.4}$$

Now, under the Equation (4.4), the moment generating function of IGW distribution can be written as

$$\begin{aligned} M(t) &= \int_0^{\infty} e^{tx}\, g(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\, E[X^r] \\ &= \frac{1}{\Gamma(\alpha)} \sum_{r=0}^{\infty} \frac{(\eta\,t\,\beta^{\frac{1}{k}})^r\;\Gamma(\alpha-\frac{r}{k})}{r!}\,, \qquad \alpha > \frac{r}{k}. \end{aligned}$$

By using appropriate moment expressions, the mean (μ), the variance (σ^2), skewness (α_3) and kurtosis (α_4) of the IGW distribution may be calcuated, respectively, as

$$\mu = E[X] = \frac{\eta\,\beta^{\frac{1}{k}}\,\Gamma(\alpha-\frac{1}{k})}{\Gamma(\alpha)}\,, \qquad \alpha > \frac{1}{k}\,.$$

$$\sigma^2 = (\eta\,\beta^{\frac{1}{k}})^2\,\left\{\frac{\Gamma(\alpha)\,\Gamma(\alpha-\frac{2}{k})\;-\;(\Gamma(\alpha-\frac{1}{k}))^2}{(\Gamma(\alpha))^2}\right\}, \quad \alpha > \frac{2}{k}$$

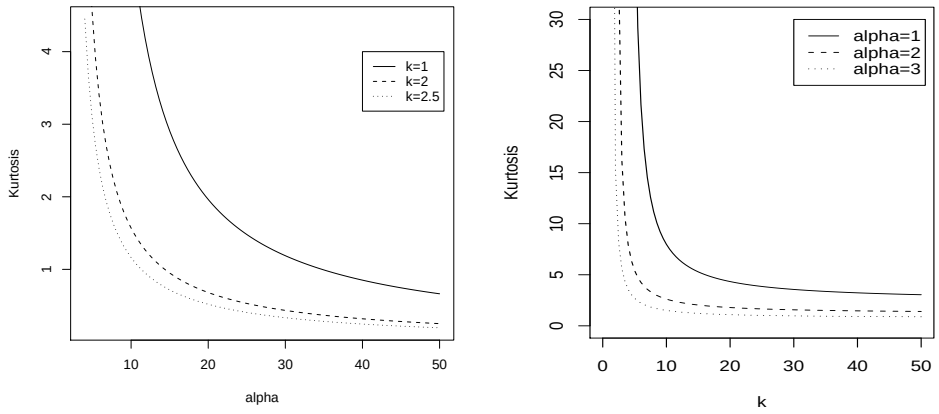
$$\alpha_3 = \frac{(\Gamma(\alpha))^2\,\Gamma(\alpha-\frac{3}{k}) - 3\,\Gamma(\alpha)\,\Gamma(\alpha-\frac{1}{k})\,\Gamma(\alpha-\frac{2}{k}) + 2\,(\Gamma(\alpha-\frac{1}{k}))^3}{\left[\Gamma(\alpha)\,\Gamma(\alpha-\frac{2}{k}) - (\Gamma(\alpha-\frac{1}{k}))^2\right]^{\frac{3}{2}}}\,, \qquad \alpha > \frac{3}{k}$$

$$\alpha_4 = \frac{(\Gamma(\alpha))^3\,\Gamma(\alpha-\frac{4}{k}) - 4\,(\Gamma(\alpha))^2\,\Gamma(\alpha-\frac{1}{k})\,\Gamma(\alpha-\frac{3}{k}) + 6\,\Gamma(\alpha)\,(\Gamma(\alpha-\frac{1}{k}))^2\,\Gamma(\alpha-\frac{2}{k})}{\left[\Gamma(\alpha)\Gamma(\alpha-\frac{2}{k}) - (\Gamma(\alpha-\frac{1}{k}))^2\right]^2}$$
$$\frac{-3(\Gamma(\alpha-\frac{1}{k}))^4}{\left[\Gamma(\alpha)\Gamma(\alpha-\frac{2}{k}) - (\Gamma(\alpha-\frac{1}{k}))^2\right]^2} \;-\; 3\,, \qquad \alpha > \frac{4}{k}.$$

Plots of the skewness and the kurtosis for some values of k as function of α and for some values of α as function of k are displayed in Figures 2a and 2b, respectively. These plots show that the skewness and kurtosis curves is decreasing, asymmetric and positive skew with k (α) for fixed α (k).

4.3. **Estimation.** We estimate the model parameters by the maximum likelihood method. Let X follows the IGW distribution with the vector of parameters $\boldsymbol{\theta} = (\alpha\,,\beta\,,k\,,\eta\,)^T$, the log-likelihood for the model parameters from a given random sample (X_1, X_2, ..., X_n) is given by

$$\ell(\boldsymbol{\theta}) = n\log(k) + n\alpha\,\log(\beta) - n\,k\alpha\log(\eta) - n\Gamma(\alpha) - (1+k\,\alpha)\sum_{i=1}^{n}\log(x_i) - \beta\,\sum_{i=1}^{n}(\frac{\eta}{x_i})^k\,.$$



(A) (B)

FIGURE 2. Kurtosis of the IGW distribution as a function of k (A) and as a function of α (B)

The components of the unit score vector $U = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial k}, \frac{\partial \ell}{\partial \eta})^T$ are

$$\frac{\partial \ell}{\partial \alpha} = n \log(\beta) + nk \log(\eta) - n\, \psi(\alpha) - k \sum_{i=1}^{n} \log(x_i) \tag{4.5}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} (\frac{\eta}{x_i})^k \tag{4.6}$$

$$\frac{\partial \ell}{\partial \eta} = \frac{n\alpha k}{\eta} - \frac{k\beta}{\eta} \sum_{i=1}^{n} (\frac{\eta}{x_i})^k \tag{4.7}$$

$$\frac{\partial \ell}{\partial k} = \frac{n}{k} + n\alpha \log(\eta) - \alpha \sum_{i=1}^{n} \log(x_i) - \beta \sum_{i=1}^{n} (\frac{\eta}{x_i})^k \log(\frac{\eta}{x_i})\,. \tag{4.8}$$

The maximum likelihood estimates (MLEs) $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{k}, \widehat{\eta})^T$ of $\boldsymbol{\theta} = (\alpha\,, \beta\,, k\,, \eta\,)^T$ is obtained by solving the nonlinear likelihood equations $\frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell}{\partial \beta} = \frac{\partial \ell}{\partial \eta} = \frac{\partial \ell}{\partial k} = 0$. These equations cannot be solved analytically and statistical software can be used to solve the equations numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the estimate $\widehat{\boldsymbol{\theta}}$. For interval estimation of $\boldsymbol{\theta} = (\alpha\,, \beta\,, k\,, \eta\,)^T$ and tests of hypotheses on these parameters we obtain the observed information matrix since the expected information matrix is very complicated and will require numerical integration.

By using Equations (4.5)-(4.8), the second derivatives may be expressed as

$$\frac{\partial^2\ell}{\partial\alpha^2} = -n\psi'(\alpha), \qquad \frac{\partial^2\ell}{\partial\alpha\partial\beta} = \frac{n}{\beta}, \qquad \frac{\partial^2\ell}{\partial\alpha\partial\eta} = \frac{nk}{\eta}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial k} = n\log(\eta) - \sum_{i=1}^{n}\log(x_i), \qquad \frac{\partial^2\ell}{\partial\beta^2} = \frac{-n\alpha}{\beta^2}, \qquad \frac{\partial^2\ell}{\partial\beta\partial\eta} = \frac{-k}{\eta}\sum_{i=1}^{n}(\frac{\eta}{x_i})^k$$

$$\frac{\partial^2\ell}{\partial\beta\partial k} = -\sum_{i=1}^{n}(\frac{\eta}{x_i})^k\,\log(\frac{\eta}{x_i}), \qquad \frac{\partial^2\ell}{\partial\eta^2} = \frac{k}{\eta^2}\Big\{-n\alpha + \beta(1-k)\sum_{i=1}^{n}(\frac{\eta}{x_i})^k\Big\}$$

$$\frac{\partial^2\ell}{\partial\eta\partial k} = \frac{1}{\eta}\Big\{n\alpha - \beta\sum_{i=1}^{n}(\frac{\eta}{x_i})^k - k\beta\sum_{i=1}^{n}(\frac{\eta}{x_i})^k\,\log(\frac{\eta}{x_i})\Big\}, \quad \frac{\partial^2\ell}{\partial k^2} = -\frac{n}{k^2} - \beta\sum_{i=1}^{n}(\frac{\eta}{x_i})^k\,\log^2(\frac{\eta}{x_i})\,.$$

The observed information matrix is given by

$$I(\boldsymbol{\theta}) = -\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha\partial\eta} & \frac{\partial^2\ell}{\partial\alpha\partial k} \\ \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\eta} & \frac{\partial^2\ell}{\partial\beta\partial k} \\ \frac{\partial^2\ell}{\partial\alpha\partial\eta} & \frac{\partial^2\ell}{\partial\beta\partial\eta} & \frac{\partial^2\ell}{\partial\eta^2} & \frac{\partial^2\ell}{\partial\eta\partial k} \\ \frac{\partial^2\ell}{\partial\alpha\partial k} & \frac{\partial^2\ell}{\partial\beta\partial k} & \frac{\partial^2\ell}{\partial\eta\partial k} & \frac{\partial^2\ell}{\partial k^2} \end{pmatrix}$$

and hence the variance covariance matrix would be $I^{-1}(\theta)$. Under regularity conditions (See Lehmann [7]) that are fulfilled for parameters in the interior of the parameter space and not on the boundary, the asymptotic distribution of $\sqrt{n}\,(\widehat{\theta}-\theta)$ is $N_4\,(0, I^{-1}(\theta)\,)$. The asymptotic normality is also useful for testing goodness-of-fit of the IGW distribution and comparing this distribution with some of its special sub-models using one of the three well-known asymptotically equivalent test statistics namely, the likelihood ratio (LR) statistic, Wald and Rao score statistics.

5. Application

In this Section, we fit IGW model to one real data set, which belongs to the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. The data set obtained from Smith and Naylor [8]. Unfortunately, the paper did not provide the units of measurement. The data set is: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. The histogram of the data along with the normal fit and

kernel fit is shown in Figure 3. As can be seen from the figure, the data is non-uniform and multimodal. In this application, we compare the IGW distribution and some their

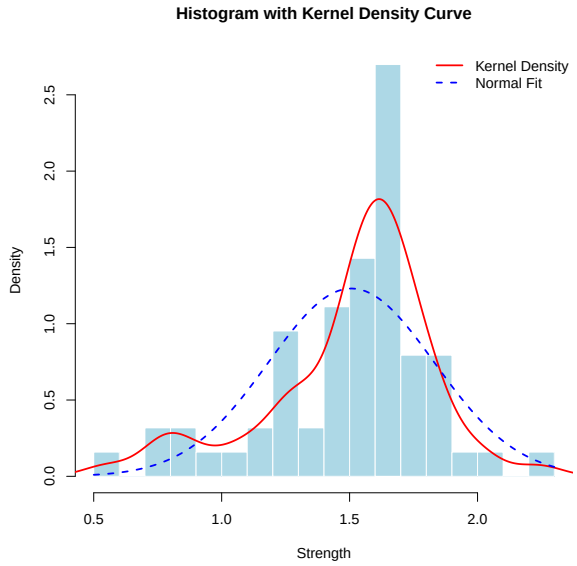


FIGURE 3. The strengths of 1.5 cm glass fibres (National Ohysical Laborotory)

sub-models. Table 1 lists the MLEs of parameters from the fitted IGW, IW, GIW, IR, IE, IM and ICS models as well as the akaike information criterion (AIC) and bayesian information criterion (BIC) values. These results indicate that the IGW model has the lowest values for these statistics among the fitted models, and therefore this model can be considered as the best model.

TABLE 1. MLEs and information criteria.

Distribution	α	β	k	η	AIC	BIC
IGW	2.5923	0.3055	1.9389	3.8494	68.5091	76.2364
IW	1	1	2.9067	1.2040	75.5267	79.3903
GIW	1	3.1469	2.9054	0.8115	77.5267	83.3221
IR	1	1	2	1.2744	86.2578	88.1897
IE	1	1	1	1.3422	142.2609	144.1927
IM	$\frac{3}{2}$	1	2	1.5608	72.3104	144.5526
ICS	1.9032	1	1	2	114.6872	116.6191

6. Conclusions

In the present work, we introduce and study a new family of distributions called the inverse gamma-F family, and derive some of its properties. In addition, we define a member of this family, namely, inverse gamma-Weibull distribution and obtain some properties of the inverse gamma-Weibull distribution; such as hazard function, quantile function, Shannon entropy, moments, order statistics, maximum likelihood estimation and the observed information matrix. We fit the inverse gamma-Weibull distribution and some of its sub-models to a real data set to compare them. The results suggest that the inveres gamma-Weibull distribution was more appropriate for data and provided a better fit compared with other sub-models.

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References

[1] Alzaatreh, A., Lee, C. and Famoye, F.(2013). A new method for generating families of continuous distributions. *Metron*, *71*, 63-79. https://doi.org/10.1007/s40300-013-0007-y

[2] Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, *81*, p.883-898. https://doi.org/10.1080/00949650903530745

[3] de Gusmao, F. R. S., Ortega, E. M. M. and Cordeiro, G. M. (2009). The generalized inverse Weibull distribution. *Statistical Papers*, *52*, pp.591-619. https://doi.org/10.1007/s00362-009-0271-3

[4] Eugene, N. C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communication in Statistics-Theory and Methods*, *31*, pp.497-512. http://dx.doi.org/10.1081/STA-120003130

[5] Gumbel, E. J. (1934). Les valeurs extremes des distributions statistiques. *Annales de l'Institut Henri Poincare*, *5*, p.115. https://www.numdam.org/item/AIHP-1935–5-2-115-0/

[6] Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Test*, *13*, pp.1-43. https://doi.org/10.1007/BF02602999

[7] Lehmann, E. L. (1983). Theory of Point Estimation. New York. John Wiley. http://dx.doi.org/10.1007/978-1-4757-2769-2

[8] Smith,R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Applied Statistics*, *36*, pp.358-369. https://doi.org/10.2307/2347795

[9] Stacy, E. W. (1962). A generalization of the Gamma distribution. *Annals of Mathematical Statistics*, *33*, pp.1187-1192. DOI: 10.1214/aoms/1177704481

[10] Zografos, K. and Balakrishnan, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, *6*, pp.344-362. https://doi.org/10.1016/j.stamet.2008.12.003

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