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SELF-DUAL CODES WITH LARGER LENGTHS OVER Z_{25}

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Abstract. In this study, new definitions of the Gray weight and the Gray map for linear codes over $R = Z_{25} + uZ_{25}$, where $u^2 = u$ are defined. Some results on self-dual codes over R are investigated. Furthermore, the structural properties of quadratic residue codes are also considered. Also two self-dual codes with parameters $[22, 11, 6]$, $[24, 12, 8]$ over Z_{25} are obtained.

1. Introduction

Let Z_{25} denote the set of integers modulo 25. A set of n-tuples over Z_{25} is called a linear code over Z_{25} or a Z_{25}-code if it is a Z_{25}-module. For a commutative ring R with identity a cyclic code C of length n over R is an ideal of $R_n = \frac{R[x]}{\langle x^n-1\rangle}$. Quadratic residue codes are a special kind of cyclic codes with prime length introduced to construct self-dual codes by adding an overall parity-check. Quadratic residue codes over finite fields have been studied extensively in the last decades. Examples of quadratic residue codes include the binary $[7, 4, 3]$ Hamming code, the binary $[23, 12, 7]$ Golay code and the ternary $[11, 6, 5]$ Golay code ([10], Ch. 6). Recently, Pless and Qian studied quadratic residue codes over Z_4 in [12]. Chiu et al. and Taeri studied the structure of quadratic residue codes over Z_8 and Z_9, respectively, (see [6] and [13]). Self-dual codes over rings have been shown to have many interesting connections to invariant theory, lattice theory and the theory

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of modular forms. For example, Bonnecaze et al. investigated the link between self-dual codes and unimodular lattices in [4]. After that self-dual codes over Z_8 and Z_9 studied in [8]. In continue a classification of self-dual codes of length $2 \leq n \leq 7$ over Z_{25} were given in [2]. So far self-dual codes over Z_{25} with large lengths have not been obtained. The detection of self-dual codes over Z_{25} with larger lengths is the motivation of this paper. The study of quadratic residue codes over the ring $R = Z_{25} + uZ_{25}$, where $u^2 = u$ is the core of this paper. The paper is organized as follows. In Section 2, we give some preliminary results and define a distance preserving Gray map from the ring R to ${Z_{25}}^2$. In Section 3, we study quadratic residue codes with lengths $p \equiv \pm 1$ and $p \equiv \pm 9$ over R. In Section 4, we give some examples of self-dual codes of large lengths over R.

2. Preliminaries

Let $R = Z_{25} + uZ_{25}$, where $u^2 = u.R$ is a commutative ring with characteristic 25, and $R \simeq \frac{Z_{25}[u]}{<u^2-u>}$. Two element u and $1-u$ are primitive idempotents. Also, each element $r \in R$ can be uniquely expressed in the form $au + b(1-u)$. The finite ring R has the following properties:

Any element $r = au + b(1-u) \in R$ is unit in R if and only if $a \not\equiv 0\,(mod\, 5)$ and $b \not\equiv 0\,(mod\, 5)$. Let A be an element of $GL_2(Z_{25})$, i.e., invertible matrix of order 2 over Z_{25}. A map $\varphi : R \to {Z_{25}}^2$ for any element $r = au + b(1-u) \in R$ is defined as:

$$\varphi(au + b(1-u)) = (a, b)A.$$

For simplicity, $(a, b)A$ is written as rA, where $r = au + b(1-u)$. Similarly, the map φ can be extended as:

$$\varphi : R^n \to {Z_{25}}^{2n}$$

$$(c_0, c_1, \ldots, c_{n-1}) \to (c_0A, c_1A, \ldots, c_{n-1}A).$$

Definition 2.1. The map φ defined above is the Gray map from R^n to ${Z_{25}}^{2n}$ corresponding to the invertible matrix A. The Lee weight of any $au + b(1-u) \in R$ is defined as: $w_L(au + b(1-u) = w_H((a, b)A)$, where w_H denotes the Hamming weight. Let C be a code of length n over R, the Lee weight of $c = (c_0, c_1, \ldots, c_{n-1}) \in C$ is defined as the sum of Lee weight of all coordinates of c. The minimum Lee weight of C is the minimum Lee weight of all codewords in C. A linear code C of length

n over R is an R-submodule of $R^n = (Z_{25} + uZ_{25})^n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two vectors of R^n. The inner product of x and y is defined as $\langle x \,.\, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where the operation is performed in R. The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in R^n \,|\, \langle x \,.\, c\rangle = 0 : \forall c \in C\}$. Code C is said to be self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$.

Theorem 2.2. *Gray map φ is a Z_{25}- linear, one to one and onto map and also distance preserving map from (R^n, Lee distance) to (${Z_{25}}^{2n}$, Hamming distance). Furthermore, let C be a self-dual code of length n over R, and let $A \in GL_2(Z_{25})$ satisfies $AA^T = \lambda I_2$, where λ is a unit in Z_{25}, A^T is the transposition of A and I_2 is the identity matrix of order 2 over Z_{25}. Then $\varphi(C)$ is a self-dual code of length $2n$ over Z_{25}.*

Proof: Let $c_1 = (c_{10}, c_{11}, \ldots, c_{1n}) \in C$ and $c_2 = (c_{20}, c_{21}, \ldots, c_{2n}) \in C^\perp$, where, for $i = 1, 2$ and $j = 0, 1, 2, \ldots, n-1$, $c_{ij} = ua_{ij} + (1-u)b_{ij}, a_{ij}, b_{ij} \in Z_{25}$. Now, from $c_1.c_2 = 0$, we have

$$\sum_{j=0}^{n-1} c_{1j}c_{2j} = u\sum_{j=0}^{n-1} a_{1j}a_{2j} + (1-u)\sum_{j=0}^{n-1} b_{1j}b_{2j} = 0.$$

Then

$$\varphi(c_1).\varphi(c_2) = (c_{10}A, c_{11}A, \ldots, c_{1n}A).(c_{20}A, c_{21}A, \ldots, c_{2n}A) = \sum_{j=0}^{n-1}(c_{1j}A)(c_{2j}A)^T = 0.$$

So $\varphi(C^\perp) \subseteq \varphi(C)^\perp$. Since $|\,\varphi(C)^\perp\,| = |\,\varphi(C^\perp)|$, then $\varphi(C^\perp) = \varphi(C)^\perp$. Note that, $C = C^\perp$ and $|\,C\,||\,C^\perp\,| = |\,R\,|^n$ shows that $dim\, C = \frac{n}{2}$. On the other hand

$$|\,\varphi(C)\,| = |\,C\,| = |\,R\,|^{\frac{n}{2}} = (25^2)^{\frac{n}{2}} = 25^n.$$

So, $dim\, \varphi(C) = \log_{25} 25^n = n$. Also,since $dim\, \varphi(C) + dim\, \varphi(C)^\perp = 2n$, then $dim\, \varphi(C)^\perp = n$. Thereby $\varphi(C)$ is a self-dual code. $\square$

For a linear code C of length n over the ring $R = Z_{25} + uZ_{25}$, let

$$C_1 = \{a \in {Z_{25}}^n \,|\, \exists\, b \in {Z_{25}}^n \,:\, au + b(1-u) \in C\}$$

and

$$C_2 = \{b \in {Z_{25}}^n \,|\, \exists a \in {Z_{25}}^n \,:\, au + b(1-u) \in C\}.$$

Clearly, C_1 and C_2 are linear code of length n over Z_{25}. Also, the linear code C can be uniquely expressed as $C = uC_1 \oplus (1-u)C_2$.

Lemma 2.3. *Let C be a linear code with lenght n over $R = Z_{25} + uZ_{25}$, then $C^{\perp} = uC_1^{\perp} \oplus (1-u)C_2^{\perp}$. Also, C is a self-dual code if and only if both C_1 and C_2 are self-dual code over Z_{25}.*

Proof: Similar to Proposition 3 *in* [9]. □

Definition 2.4. Let C be a code of length n over R and $P(C)$ be its polynomial representation, i.e.$P(C) = \{\Sigma_{i=0}^{n-1} c_i x^i \mid (c_0, c_1, c_2, \ldots, c_{n-1}) \in C\}$. A linear code C of length n over R is a cyclic code if and only if $P(C)$ is an ideal of the ring $R_n = \frac{R[x]}{\langle x^n - 1\rangle}$. The ideal $P(C)$ is called the ideal corresponding to code C.

Note that, a linear code $C = uC_1 \oplus (1-u)C_2$ is a cyclic code over $R = Z_{25} + uZ_{25}$ if and only if C_1 and C_2 are both cyclic code over Z_{25}.

Theorem 2.5. (*Theorem* 3.4 *in* [11]) *Suppose p is a prime not dividing n and C is a cyclic Z_{p^m}-code. Then there exist a collection of pairwise-coprime polynomials $F_0, F_1, \ldots, F_m$ such that $F_0 F_1 \ldots F_m = x^n - 1$ and $C = \langle \hat{F_1}, p\hat{F_2}, \ldots, p^{m-1}\hat{F_m}\rangle$, where $\hat{F_i} = \frac{x^n-1}{F_i}$, for $i = 1, 2, \ldots m$.*□

An element $e(x) \in R_n$ satisfying $e^2(x) = e(x)$ is called an idempotent. Equivalently, as polynomials $e^2(x) \equiv e(x)(mod\,(x^n - 1))$. Each cyclic code over R contains a unique idempotent, which generates the ideal. This idempotent is called the generating idempotent of the cyclic code.

Theorem 2.6. (i) *Let C be a cyclic code of length n over a finite ring R generated by the idempotent $e(x)$ in quetiont ring $\frac{R[x]}{\langle x^n-1\rangle}$, then $C^{\perp}$ is generated by the idempotent $1 - e(x^{-1})$.*
(ii) *Let C_1 and C_2 be cyclic codes of length n over a finite ring R generated by the idempotents $e_1(x)$, $e_2(x)$ in $\frac{R[x]}{\langle x^n-1\rangle}$, respectively. Then $C_1 \cap C_2$ and $C_1 + C_2$ are generated by the idempotents $e_1(x)e_2(x)$ and $e_1(x) + e_2(x) - e_1(x)e_2(x)$, respectively.*

Proof: Similar to Theorem 7 *in* [12].□

Let C be a cyclic code over Z_{25}, then by Theorem 2.5, there exist uniqe monic polynomials $f(x), g(x), h(x) \in Z_5[x]$, such that $x^n - 1 = f(x)h(x)g(x)$ and $C = \langle f(x)g(x), 5f(x)h(x)\rangle$.

Lemma 2.7. *Let $C = uC_1 \oplus (1-u)C_2$ be a cyclic code of length n over $R = Z_{25} + uZ_{25}$, then $C = \langle uf_1(x)g_1(x) + (1-u)f_2(x)g_2(x),\ 5uf_1(x)h_1(x) + 5(1-u)f_2(x)h_2(x)\rangle$, where $x^n - 1 =$*

$f_1(x)h_1(x)g_1(x) = f_2(x)h_2(x)g_2(x)$, *and for* $i = 1, 2$, $C_i = \langle f_i(x)g_i(x), 5f_i(x)h_i(x)\rangle$ *is a cyclic code over* Z_{25}.

Proof: Let $\overline{C} = \langle uf_1(x)g_1(x) + (1-u)f_2(x)g_2(x), 5uf_1(x)h_1(x) + 5(1-u)f_2(x)h_2(x)\rangle$.

Also, let $C_1 = \langle f_1(x)g_1(x), 5f_1(x)h_1(x)\rangle$,*and* $C_2 = \langle f_2(x)g_2(x), 5f_2(x)h_2(x)\rangle$.

Clearly $\overline{C} \subseteq C$, *and hence* $uC_1 = u\overline{C}$, $(1-u)C_2 = (1-u)\overline{C}$. *This implies that* $uC_1 \oplus (1-u)C_2 \subseteq \overline{C}$. *Thus* $C = \overline{C}$.$\square$

Corollary 2.8. *Let* $R = Z_{25} + uZ_{25}$, *then* $\frac{R[X]}{\langle x^n-1\rangle}$ *is a principal ideal ring.*

Proof: By notations Lemma 2.7, *Let* $w(x) = f(x)g(x) + 5f(x)h(x)$. *Similar to Theorem* 3.6 *in* [7], *we can prove that* $C = \langle w(x)\rangle$.$\square$

Note that, the number of distinct cyclice codes of length n over $R = Z_{25} + uZ_{25}$ is 25^r, where r is number of the basic irreducible factors of $x^n - 1$ over Z_{25}. Now, Let $f(x) \in Z_{25}[x]$, be a polynomial of degree k, then $f^\star(x) = x^k f(x^{-1})$ will be denote its reciprocal polynomial. Note that, $(f(x)g(x))^\star = f^\star(x)g^\star(x)$ for $f(x)$, $g(x) \in Z_{25}[x]$. In fact, $(f(x)g(x))^\star = f^\star(x)g^\star(x)$ for $f(x)$, $g(x) \in \frac{Z_{25}[x]}{\langle x^n-1\rangle}$, provided $deg(f(x)g(x)) < n$.

Lemma 2.9. *Let* $C = \langle f(x)g(x), 5f(x)h(x)\rangle$ *be a cyclic code with odd length* n *over* Z_{25}, *where* $f(x), g(x)$ *and* $h(x)$ *are monic polynomials such that* $f(x)h(x)g(x) = x^n - 1$. *Then* C *is self-dual code if and only if* $f(x) = h^\star(x)$ *and* $g(x) = g^\star(x)$.

Proof: The proof is similar to proof of Theorem 12 .3 .20 *in* [10] *for cyclic codes over* Z_4.$\square$

Lemma 2.10. *Let* $C = \langle uf_1(x)g_1(x)+(1-u)f_2(x)g_2(x), 5uf_1(x)h_1(x)+5(1-u)f_2(x)h_2(x)\rangle$ *be a cyclic code over* $R = Z_{25} + uZ_{25}$, *where* $x^n - 1 = f_1(x)h_1(x)g_1(x) = f_2(x)h_2(x)g_2(x)$ *and for* $i = 1, 2$, $C_i = \langle f_i(x)g_i(x), 5f_i(x)h_i(x)\rangle$ *is a cyclic code over* Z_{25}. *Then* C *is self-dual if and only if* $f_2(x) = h_2^*(x)$, $g_1(x) = g_1^*(x)$ *and* $f_1(x) = h_1^*(x)$, $g_2(x) = g_2^*(x)$.

Proof: Since $C^\perp = uC_1^\perp \oplus (1-u){C_2}^\perp$, *then* $C^\perp$ *is cyclic code if and only if* C *is a cyclic code. Also by Lemma* 2.4, *code* C *is self-dual over* $R = Z_{25} + uZ_{25}$ *if and only codes* C_1 *and* C_2 *are both self-dual over* Z_{25}. *Now, by Lemma* 2.11, *the proof is compelete.*$\square$

Since Z_{25} is a chain ring with unique maximal ideal $\langle 5\rangle$, by Theorem 4.4 in [3], we have the following lemma.

Lemma 2.11. *Non-trivial cyclic self-dual codes of length n over Z_{25} exist if and only if $5^i \neq -1(mod\, n)$ for all positive integer i.*

Lemma 2.12. *Let C be a cyclic code of length n, over the ring $R = Z_{25} + uZ_{25}$, and $\gcd(n, 25) = 1$, then there exists a unique idempotent element $e(x) = ue_1(x) + (1 - u)e_2(x) \in R[x]$ such that $C = \langle e(x) \rangle$.*

Proof: *Since $\gcd(n, 25) = 1$, by Theorem 5.1 in [11], there exist unique idempotent elements $e_1(x), e_2(x) \in Z_{25}[x]$, such that $C_1 = \langle e_1(x) \rangle$, $C_2 = \langle e_2(x) \rangle$. Then $C = \langle ue_1(x) + (1 - u)e_2(x) \rangle$, let $e(x) = ue_1(x) + (1 - u)e_2(x)$. Then $e^2(x) = ue_1^2(x) + (1 - u)e_2^2(x) = ue_1(x) + (1 - u)e_2(x) = e(x)$. So $e(x)$ is an idempotent of code C. If there exists another $d(x) \in C$, such that $C = \langle d(x) \rangle$, then $d(x) \in C = (e(x))$, thereby $d(x) = a(x)e(x)$. Then $d(x)e(x) = a(x)e^2(x) = a(x)e(x)$ and hence $d(x) = e(x)$, which implies that $e(x)$ is unique.*□

Lemma 2.13. *Let $C = uC_1 \oplus (1 - u)C_2$ be a cyclic code of length n over $R = Z_{25} + uZ_{25}$. Let $e(x) = ue_1(x) \oplus (1 - u)e_2(x)$, where for $i = 1, 2$, $e_i(x)$ is generating idempotent of C_i over Z_{25}. Then $1 - e(x^{-1})$ is the generating idempotent for dual code $C^{\perp}$.*

Proof: *Remember that $C^{\perp} = u{C_1}^{\perp} \oplus (1 - u){C_2}^{\perp}$ and $C^{\perp}$ is a cyclic code if and only if $C_1^{\perp}$, $C_2^{\perp}$ are both cyclic codes. By Theorem 2.7, we have $C_i^{\perp} = \langle 1 - e_i(x^{-1}) \rangle$, for $i = 1, 2$. By Lemma 2.14, we have $u(1 - e_1(x^{-1}) + (1 - u)(1 - e_2(x^{-1}) = 1 - e(x^{-1})$ is generating idempotent for code $C^{\perp}$.*□

3. Quadratic Residue Codes Over $R = Z_{25} + uZ_{25}$.

Quadratic residue codes are duadic codes over Z_q of odd prime length $n = p$, where q is a power of a prime number and q must be a square modulo n. We will let $n = p$ be an odd prime not dividing q, we will assume that q is a prime power that is a square modulo p. Let Q_p denote the set of nonzero squares modulo p and let N_p be the set of nonsquares modulo p. Let $Q(x) = \Sigma_{i \in Q_p} x^i$, $N(x) = \Sigma_{i \in N_p} x^i$ and $h(x) = 1 + Q(x) + N(x)$.

Theorem 3.1. *The Legendre symbol $(\frac{5}{p}) = 1$ if and only if $p \equiv \pm 1\, (mod\, 20)$ and $p \equiv \pm 9\, (mod\, 20)$.*

Proof: *See Theorem 1.1 in [1].*□

By Theorem 3.1 for considering quadratic residue code over Z_5(and hence over Z_{25}), we must assume that $p \equiv \pm 1\, (mod\, 20)$ and $p \equiv \pm 9\, (mod\, 20)$. By the introducing of *quadratic residue codes over Z_{25} in [1], we now discuss the quadratic residue codes over $R = Z_{25} + uZ_{25}$. We assume $e_1(x)$ and $e_2(x)$ be generating idempotent of quadratic residue codes C_1, C_2, respectively. Then $e(x) = ue_1(x) + (1-u)e_2(x)$ is a generating idempotent for code $C = uC_1 \oplus (1-u)C_2$.*

By Theorem 2.7 in [1] and Lemma 2.14, we have the following definition.

Definition 3.2. Suppose that $p = 20k + 1$, then

(i) If $k = 5t$, let $D_1 = \langle u(1 + N(x)) + (1-u)(1 + Q(x))\rangle$,
$D_2 = \langle u(1 + Q(x)) + (1-u)(1 + N(x))\rangle$,
$E_1 = \langle 24uQ(x) + (1-u)(24N(x))\rangle$,
$E_2 = \langle 24uN(x) + (1-u)(24Q(x))\rangle$.

(ii) If $k = 5t+1$, let $D_1 = \langle u(20Q(x) + 11N(x) + 16) + (1-u)(11Q(x) + 20N(x) + 16)\rangle$,
$D_2 = \langle u(11Q(x) + 20N(x) + 16) + (1-u)(20Q(x) + 11N(x) + 16)\rangle$,
$E_1 = \langle u(14Q(x) + 5N(x) + 10) + (1-u)(5Q(x) + 14N(x) + 10)\rangle$,
$E_2 = \langle u(5Q(x) + 14N(x) + 10) + (1-u)(14Q(x) + 5N(x) + 10)\rangle$,

(iii) If $k = 5t+2$, let $D_1 = \langle u(15Q(x) + 21N(x) + 6) + (1-u)(21Q(x) + 15N(x) + 6)\rangle$,
$D_2 = \langle u(21Q(x) + 15N(x) + 6) + (1-u)(15Q(x) + 21N(x) + 6)\rangle$,
$E_1 = \langle u(4Q(x) + 10N(x) + 20) + (1-u)(10Q(x) + 4N(x) + 20)\rangle$,
$E_2 = \langle u(10Q(x) + 4N(x) + 20) + (1-u)(4Q(x) + 10N(x) + 20)\rangle$.

(iv) If $k = 5t+3$, let $D_1 = \langle u(6Q(x) + 10N(x) + 21) + (1-u)(10Q(x) + 6N(x) + 21))\rangle$,
$D_2 = \langle u(10Q(x) + 6N(x) + 21) + (1-u)(6Q(x) + 10N(x) + 21)\rangle$,
$E_1 = \langle u(19Q(x) + 15N(x) + 5) + (1-u)(15Q(x) + 19N(x) + 5)\rangle$,
$E_2 = \langle u(15Q(x) + 19N(x) + 5) + (1-u)(19Q(x) + 15N(x) + 5)\rangle$.

(v) If $k = 5t+4$, let $D_1 = \langle u(5Q(x) + 16N(x) + 11) + (1-u)(16Q(x) + 5N(x) + 11)\rangle$,
$D_2 = \langle u(16Q(x) + 5N(x) + 11) + (1-u)(5Q(x) + 16N(x) + 11)\rangle$,
$E_1 = \langle u(9Q(x) + 20N(x) + 15) + (1-u)(20Q(x) + 9N(x) + 15)\rangle$,
$E_2 = \langle u(20Q(x) + 9N(x) + 15) + (1-u)(9Q(x) + 20N(x) + 15)\rangle$.

These twenty cyclic codes are called the quadratic residue codes over $Z_{25} + uZ_{25}$. Now, Let a be an integer such that $\gcd(a, n) = 1$, the function μ_a defined on $\{0, 1, \ldots, n-1\}$ by $\mu_a(i) \equiv ia(mod\, n)$ is a permutation of the coordinate positions $\{0, 1, \ldots, n-1\}$ of a cyclic code of length n and is called a multiplier. This map acts on any polynomials $f(x) = \Sigma c_i x^i \in R[x]$ as $\mu_a(\Sigma c_i x^i) = \Sigma c_i x^{ia}$.

Theorem 3.3. *Let $p = 20k + 1$, then the following conditions on quadratic residue codes does hold.*
(i) If $a \in Q_p$, then $\mu_a(D_i) = D_i$ and $\mu_a(E_i) = E_i$. If $a \in N_p$, then $\mu_a(D_i) = D_j$ and $\mu_a(E_i) = E_j$, for $i, j \in \{1, 2\}$ and $i \neq j$.
(ii) $D_1 \cap D_2 = \langle l(x)\rangle$ and $D_1 + D_2 = R_p$, where $l(x)$ is a suitable element of $\{h(x), 6h(x), 11h(x)$ $, 16h(x), 21h(x)\}$.
(iii) $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \langle l(x)^{\perp}\rangle$.
(iv) For $i = 1, 2$, we have $D_i = E_i + \langle l(x)\rangle$.
(v) For $i = 1, 2$, we have $\mid D_i \mid = 25^{p+1}$ and $\mid E_i \mid = 25^{p-1}$.
(vi) $E_1^{\perp} = D_2$ and $E_2^{\perp} = D_1$.

Proof: (i) Let $p = 20k + 1$, we prove only the case $k = 5t$, other cases are proved similarly. In this case $l(x) = h(x)$. If $a \in N_p$, then $\mu_a(u(24Q(x)) + (1 - u)(24N(x))) = u(24N(x)) + (1 - u)(24Q(x))$. This shows that $\mu_a(E_1) = E_2$. Similarly, we can show that $\mu_a(E_2) = E_1$ and $\mu_a(D_i) = D_j$, for $i, j \in \{1, 2\}$ and $i \neq j$.

(ii) By Theorem 2.7, $D_1 \cap D_2 = \langle (u(1 + Q(x)) + (1 - u)(1 + N(x)))(u(1 + N(x)) + (1 - u)(1 + Q(x)))\rangle$.

Since $u(1+N(x))+(1-u)(1+Q(x))+u(1+Q(x))+(1-u)(1+N(x)) = 1+h(x)$, then

$(u(1 + N(x)) + (1 - u)(1 + Q(x)))h(x) = (u(1 + N(x)) + (1 - u)(1 + Q(x)))(24 + 1 + h(x)) = 24(u(1 + N(x))) + (1 - u)(1 + Q(x))) + u(1 + N(x))^2 + u(1 + N(x))(1 + Q(x)) + (1 - u)(1 + Q(x))^2 + (1 - u)(1 + N(x))(1 + Q(x)) = (u(1 + N(x)) + (1 - u)(1 + Q(x)))(u(1 + Q(x)) + (1 - u)(1 + N(x)))$.

Since $p = 20(5t) + 1 \equiv 1 \,(mod\, 25)$, then $\frac{p-1}{2} \equiv 0 \,(mod\, 25)$, thereby

$(u(1+N(x)) + (1-u)(1+Q(x)))(u(1+Q(x)) + (1-u)(1+N(x))) = (uQ(x) + N(x) - uN(x) + 1)h(x) = u(\frac{p-1}{2})h(x) + (\frac{p-1}{2})h(x) - u(\frac{p-1}{2})h(x) + h(x) = h(x)$. *This shows that $D_1 \cap D_2 = \langle h(x)\rangle$. Again, by Theorem 2.7,*

$u(1+N(x)) + (1-u)(1+Q(x)) + u(1+Q(x)) + (1-u)(1+N(x)) - (u(1+N(x)) + (1-u)(1 + Q(x)))(u(1 + Q(x)) + (1 - u)(1 + N(x)))$ *is a generating idempotent for $D_1 + D_2$. This shows that $D_1 + D_2 = R_p$.*

(iii) By Theorem 2.7, $E_1 \cap E_2 = \langle (24uQ(x)+24(1-u)N(x))(24uN(x)+(1-u)(24Q(x)) \rangle$. *As* $24uQ(x) + 24(1-u)N(x) + 24uN(x) + 24(1-u)Q(x) = 1 - h(x)$. *Also*

$(24uQ(x) + 24(1-u)N(x))(-h(x)) = (24uQ(x) + 24(1-u)N(x))(24 + 1 - h(x) = (24uQ(x) + 24(1-u)N(x)) + (24uQ(x) + 24(1-u)N(x))(24uQ(x) + 24(1-u)N(x) + 24uN(x) + 24(1-u)Q(x)) = (24uQ(x) + 24(1-u)N(x))(24uN(x) + (1-u)(24Q(x))$.

Since $\frac{p-1}{2} \equiv 0 \,(mod\, 25)$, *then* $(24uQ(x) + 24(1-u)N(x)(-h(x)) = u(\frac{p-1}{2})h(x) + (\frac{p-1}{2})(h(x)) - u(\frac{p-1}{2})(h(x)) = 0$. *This shows that* $E_1 \cap E_2 = \{0\}$. *Again, by Theorem* 2.7, *we know that*

$24uQ(x)+24(1-u)N(x)+24uN(x)+24(1-u)Q(x)-(24uQ(x)+24(1-u)N(x))(24uN(x)+(1-u)(24Q(x))$, *is a generating idempotent for code* $E_1 + E_2$. *This shows that* $E_1 + E_2 = \langle 1 - h(x) \rangle = \langle h(x) \rangle^{\perp}$.

(iv) Theorem 2.7 *shows that,* $E_1 + \langle l(x) \rangle$ *has idempotent generator*

$$24uQ(x) + 24(1-u)N(x) + h(x) - (24uQ(x) + 24(1-u)N(x))h(x).$$

Note that, $(24uQ(x) + 24(1-u)N(x))(-h(x)) = 0$. *Then* $24uQ(x) + 24(1-u)N(x) + h(x) = u(1 + N(x)) + (1-u)(1 + Q(x))$. *Therefore* $E_1 + \langle l(x) \rangle = D_1$. *Similarly, we can show that* $E_2 + \langle l(x) \rangle = D_2$.

(v) Since $D_1 + D_2 = R_p$ *and* D_1, D_2 *are equivalent, then we must have* $25^{2p} = |\,D_1 + D_2\,| = \frac{|\,D_1\,|\,|\,D_2\,|}{|\,D_1 \cap D_2\,|}$. *Since* $|\,D_1 \cap D_2\,| = 25^2$, *then* $|\,D_1\,| = |\,D_2\,| = 25^{p+1}$. *Also,*$D_1 = E_1 + \langle l(x) \rangle$ *and* $(24uQ(x) + (1-u)(24N(x)))h(x) = 0$, *this shows that* $|\,E_1\,| = 25^{p-1}$.

Similarly, we can show that $|\,E_2\,| = 25^{p-1}$.

(vi) As $-1 \in Q_p$, *by Theorem* 2.7, *the generating idempotent of* ${E_1}^{\perp}$ *is*

$$1 - \mu_{-1}(24uQ(x) + (1-u)(24N(x))) = u(1 + Q(x)) + (1-u)(1 + N(x)) = D_2.$$

Then ${E_1}^{\perp} = D_2$. *Similarly, we can show that* ${E_2}^{\perp} = D_1$.□

By Theorem 2.8 in [1] and Lemma 2.14, we have the following definition.

Definition 3.4. Suppose that $p = 20k - 1$, then
(i) If $k = 5t$, let $D_1 = \langle 24uN(x) + (1-u)(24Q(x)) \rangle$,
$D_2 = \langle 24uQ(x) + (1-u)(24N(x)) \rangle$,

$E_1 = \langle u(1+Q(x)) + (1-u)(1+N(x))\rangle$,
$E_2 = \langle u(1+N(x)) + (1-u)(1+Q(x))\rangle$.

(*ii*) If $k = 5t+1$, let $D_1 = \langle u(9Q(x)+20N(x)+15) + (1-u)(20Q(x)+9N(x)+15)\rangle$,
$D_2 = \langle u(20Q(x)+9N(x)+15) + (1-u)(9Q(x)+20N(x)+15)\rangle$,
$E_1 = \langle u(5Q(x)+16N(x)+11) + (1-u)(16Q(x)+5N(x)+11)\rangle$,
$E_2 = \langle u(16Q(x)+5N(x)+11) + (1-u)(5Q(x)+16N(x)+11)\rangle$.

(*iii*) If $k = 5t+2$, let $D_1 = \langle u(19Q(x)+15N(x)+5) + (1-u)(15Q(x)+19N(x)+5)\rangle$,
$D_2 = \langle u(15Q(x)+19N(x)+5) + (1-u)(19Q(x)+15N(x)+5)\rangle$,
$E_1 = \langle u(10Q(x)+6N(x)+21) + (1-u)(6Q(x)+10N(x)+21)\rangle$,
$E_2 = \langle u(6Q(x)+10N(x)+21) + (1-u)(10Q(x)+6N(x)+21)\rangle$.

(*iv*) If $k = 5t+3$, let $D_1 = \langle u(4Q(x)+10N(x)+20) + (1-u)(10Q(x)+4N(x)+20))\rangle$,
$D_2 = \langle u(10Q(x)+4N(x)+20) + (1-u)(4Q(x)+10N(x)+20)\rangle$,
$E_1 = \langle u(15Q(x)+21N(x)+6) + (1-u)(21Q(x)+15N(x)+6)\rangle$,
$E_2 = \langle u(21Q(x)+15N(x)+6) + (1-u)(15Q(x)+21N(x)+6)\rangle$.

(*v*) If $k = 5t+4$, let $D_1 = \langle u(14Q(x)+5N(x)+10) + (1-u)(5Q(x)+14N(x)+10)\rangle$,
$D_2 = \langle u(5Q(x)+14N(x)+10) + (1-u)(14Q(x)+5N(x)+10)\rangle$,
$E_1 = \langle u(20Q(x)+11N(x)+16) + (1-u)(11Q(x)+20N(x)+16)\rangle$,
$E_2 = \langle u(11Q(x)+20N(x)+16) + (1-u)(20Q(x)+11N(x)+16)\rangle$.

This cyclic codes of length p are called the quadratic residue codes over $R = Z_{25} + Z_{25}$. Similar to Theorem 3.3, we have the same result.

Theorem 3.5. *Let $p = 20k-1$, then the following conditions on quadratic residue codes does hold.*
(i) If $a \in Q_p$, then $\mu_a(D_i) = D_i$ and $\mu_a(E_i) = E_i$. If $a \in N_p$, then $\mu_a(D_i) = D_j$ and $\mu_a(E_i) = E_j$, for $i, j \in \{1,2\}$ and $i \neq j$.
(ii) $D_1 \cap D_2 = \langle l(x)\rangle$ and $D_1 + D_2 = R_p$, where $l(x)$ is suitable element of $\{-h(x), 4h(x), 9h(x), 14h(x), 19h(x)\}$.
(iii) $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \langle l(x)^{\perp}\rangle$.
(iv) For $i = 1, 2$, we have $D_i = E_i + \langle l(x)\rangle$.
(v) For $i = 1, 2$, we have $|\, D_i \,| = 25^{p+1}$ and $|\, E_i \,| = 25^{p-1}$.
(vi) E_1, E_2 are self-orthogonal code and for $i \in \{1,2\}$ we have, $E_i^{\perp} = D_i$.

Proof: We only need to prove part (iv), the proof of other parts are similar to Theorem 3.3, so we omit it. Let $k = 5t$, note that $-1 \in N_p$ and E_1 has the idempotent generator

$$1 - \mu_{-1}(u(1 + Q(x) + (1 - u)(1 + N(x))) = u(-N(x)) + (1 - u)(-Q(x)).$$

Then ${E_1}^{\perp} = D_2$. Similarly, we can show that ${E_2}^{\perp} = D_1$.□

The proof of the following theorem is similar to Theorem 3.3 and 3.5, so we omit it.

Theorem 3.6. *Let $p = 20k \pm 9$, then the following conditions on quadratic residue codes does hold.*
(i) If $a \in Q_p$, then $\mu_a(D_i) = D_i$ and $\mu_a(E_i) = E_i$. If $a \in N_p$, then $\mu_a(D_i) = D_j$ and $\mu_a(E_i) = E_j$, for $i, j \in \{1, 2\}$ and $i \neq j$.
(ii) $D_1 \cap D_2 = \langle l(x) \rangle$ and $D_1 + D_2 = R_p$, where $l(x)$ is suitable element of $\{14h(x), 19h(x), -h(x), 4h(x), 9h(x)\}$, if $p = 20k + 9$ and $l(x)$ is suitable element of $\{16h(x), 21h(x), h(x), 6h(x), 11h(x)\}$, if $p = 20k + 11$.
(iii) $E_1 \cap E_2 = \{0\}$ and $E_1 + E_2 = \langle l(x)^{\perp} \rangle$.
(iv) For $i = 1, 2$, we have $D_i = E_i + \langle l(x) \rangle$.
(v) For $i = 1, 2$, we have $|\, D_i \,| = 25^{p+1}$ and $|\, E_i \,| = 25^{p-1}$.
(vi) If $p = 20k + 9$, then $E_1^{\perp} = D_2$ and $E_2^{\perp} = D_1$. If $p = 20k + 11$, then two codes E_1, E_2 are self-orthogonal and for $i \in \{1, 2\}$ we have $E_i^{\perp} = D_i$.

Definition 3.7. The extended code of a quadratic residue code C over Z_{25} denoted by $\bar{C}$, which is the code obtained by adding a specific column to the generator matrix of C. In other words extension $\bar{C}$ of C is defined by $\bar{C} = \{\bar{c} \,|\, c \in C\}$, where $\bar{c} = (c_\infty, c_o, c_1, \ldots, c_{p-1})$, $c_\infty + c_0 + c_1 + \cdots + c_{p-1} \equiv 0 \,(mod\, 25)$.

Let $p = 20k + 1$ we define $\tilde{D}_1$ to be the Z_{25}-code generated by the matrix

$$\begin{pmatrix} \infty & 0 & 1 & 2 & \cdots & p-1 \\ 0 & & & & & \\ 0 & & & G_1 & & \\ . & & & & & \\ . & & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where each row of G_1 is a cyclic shift of the $-Q(x)$ when $k = 5t$, is a cyclic shift of the $14Q(x) + 5N(x) + 10$ when $k = 5t + 1$, is a cyclic shift of the $4Q(x) + 10N(x) + 20$ when $k = 5t + 2$, is a cyclic shift of the $19Q(x) + 15N(x) + 5$ when $k = 5t + 3$, is a cyclic shift of the $9Q(x) + 20N(x) + 15$ when $k = 5t + 4$. Similarly we define $\tilde{D}_2$.

Theorem 3.8. *(i) Let $p = 20k - 1$ and D_1, D_2 are quadratic residue codes over R also $\bar{D}_1, \bar{D}_2$ denote their extended codes, then $\bar{D}_1, \bar{D}_2$ are self-dual codes.*
(ii) Let $p = 20k + 1$, and D_1, D_2 are quadratic residue codes over R, then ${\bar{D}_1}^{\perp} = \tilde{D}_2$ and ${\bar{D}_2}^{\perp} = \tilde{D}_1$.

Proof: (i) We only prove the case $k = 5t + 1$, other cases are proved similarly. By Theorem 3.5, we have $D_1 = E_1 + \langle 4h \rangle$. Also, $\bar{D}_1$ has the following generator matrix:

$$\begin{pmatrix} \infty & 0 & 1 & 2 & \cdots & p-1 \\ 0 & & & & & \\ 0 & & & G_1 & & \\ . & & & & & \\ . & & & & & \\ 24 & 4 & 4 & 4 & \cdots & 4 \end{pmatrix},$$

where each row of G_1 is a cyclic shift of the $5Q(x) + 16N(x) + 11$. Since G_1 is a generator matrix for code E_1 and E_1 is self-orthogonal (Theorem 3.5 (vi)), the rows of G_1 are orthogonal to each other and also orthogonal to $4h$ (Theorem 3.5(iii)). We know that the vector $(24, 4h)$ is orthogonal to itself. This shows that $\bar{D}_1$ is self-orthogonal. Since $|\bar{D}_1^{\perp}| = |R|^{p+1} - |\bar{D}_1| = |\bar{D}_1|$, then $\bar{D}_1$ is a self-dual code. Similarly, we can show that $\bar{D}_2$ is a self-dual code.
(ii) We prove only the case $k = 5t + 2$ the other cases are proved similarly. Note that, in this case $D_1 = E_1 + \langle 11h \rangle$, by Theorem 3.3 (iv). Then $\bar{D}_1$ has the following generator matrix:

$$\begin{pmatrix} \infty & 0 & 1 & 2 & \cdots & p-1 \\ 0 & & & & & \\ 0 & & & G_1 & & \\ . & & & & & \\ . & & & & & \\ 24 & 11 & 11 & 11 & \cdots & 11 \end{pmatrix},$$

where each row of G_1 *is a cyclic shift of the* $4Q(x)+10N(x)+20$. *By Theorem* 3.3 (*vi*), ${E_1}^{\perp} = D_2$ *and* G_1 *generate* E_1. *Since the product of the vectors* $(24, 11, \ldots, 11)$ *and* $(1, 1, \ldots, 1)$ *is* $24 + 11\,p \equiv 0\,(mod\,25)$, *then any row in the above matrix is orthogonal to any row in the matrix which defines* $\tilde{D}_2$. *Then* $\tilde{D}_2 \subseteq \bar{D_1}^{\perp}$. *Since* $|\tilde{D}_2| = |\bar{D_1}^{\perp}| = 25^{p+1}$, *we must have* $\bar{D_1}^{\perp} = \tilde{D}_2$. *Similarly, we can show that* $\bar{D_2}^{\perp} = \tilde{D}_1$.□

The proof of the two following theorems is similar to Theorem 3.8, so we omit it.

Theorem 3.9. (*i*) *Let* $p = 20k + 11$ *and* D_1, D_2 *are quadratic residue codes over* R *and* $\bar{D}_1$, $\bar{D}_2$ *denote their extended codes. Then* $\bar{D}_1$, $\bar{D}_2$ *are self-dual codes.*
(*ii*) *If* $p = 20k + 9$ *and* D_1, D_2 *are quadratic residue codes over* R, *then* $\bar{D_1}^{\perp} = \tilde{D}_2$ *and* $\bar{D_2}^{\perp} = \tilde{D}_1$.

4. Numerical examples

In this section, some examples are given to illustrate the main work in this manuscript. Let $M = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$ be a matrix of $GL_2(Z_{25})$. Clearly $MM^t = 8I_2$. Suppose that C is a self-dual code of length n over the ring $R = Z_{25} + uZ_{25}$ and φ be the Gray map corresponding to matrix M. Theorem 2.2, shows that $\varphi(C)$ is a self-dual code of length $2n$ over ring Z_{25}.

Example 1. Since $5^j \neq -1\,(mod\,11)$, for any positive integer j. Then Lemma 2.11, shows that there exists a self-dual code of length 11 over ring R. Note that $x^{11} - 1 = (x+24)(x^5+17x^4+24x^3+x^2+16x+24)(x^5+9x^4+24x^3+x^2+8x+24)$ over $Z_{25}[x]$. Now, let $g(x) = 1-x$ and $f(x) = x^5+17x^4+24x^3+x^2+16x+24$, then $f^{\star}(x) = -(x^5+9x^4+24x^3+x^2+8x+24)$. Therefore $x^{11}-1 = g(x)f(x)f^{\star}(x)$. Let $C_1 = C_2 = \langle f^{\star}(x)g(x), 5f(x)f^{\star}(x)\rangle$. By Lemma 2.10, code $C = \langle f^{\star}(x)g(x), 5f(x)f^{\star}(x)\rangle$ is a cyclic self-dual code over the ring $R = Z_{25}+uZ_{25}$. Theorem 2.2, shows that $\varphi(C)$ is a cyclic self-dual code of length 22 over Z_{25}. The image of code C under Gray map φ is a code of dimension 11 with minimum Hamming weight 6.

Example 2. Let $p = 11$. We considere the quadratic residue codes of length 11 over $R = Z_{25} + uZ_{25}$. Let Q_{11} denote the set of quadratic residue modulo 11 and N_{11} the set of non residue modulo 11. So, $Q_{11} = \{1, 3, 4, 5, 9\}$ and $N_{11} = \{2, 6, 7, 8, 10\}$. Let

$$Q(x) = \sum_{i \in Q_{11}} x^i,\ N(x) = \sum_{j \in N_{11}} x^j.$$

Since $11 = 20k + 11$, by Theorem 2.10 in [1], we have

$$D_1 = \langle u(22Q(x) + 19N(x) + 21) + (1-u)(19Q(x) + 22N(x) + 21)\rangle,$$

$$D_2 = \langle u(19Q(x) + 22N(x) + 21) + (1-u)(22Q(x) + 19N(x) + 21)\rangle,$$

$$E_1 = \langle u(6Q(x) + 3N(x) + 5) + (1-u)(3Q(x) + 6N(x) + 5)\rangle,$$

$$E_2 = \langle u(3Q(x) + 6N(x) + 5) + (1-u)(6Q(x) + 3N(x) + 5)\rangle,$$

are quadratic residue codes of length 11 over the ring $R = Z_{25} + uZ_{25}$. Two codes E_1 and E_2 have the following Z_{25}-generator matrices respectively.

$$G_1 = \begin{pmatrix} uA_{1,1} \\ (1-u)A_{1,2} \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} uA_{2,1} \\ (1-u)A_{2,2} \end{pmatrix},$$

where $A_{1,1} = A_{2,2} = [I_5 \mid B]$ and $A_{1,2} = A_{2,1} = [I_{10} \mid B'^T]$. Also, B and B' are the following matrices.

$$B = \begin{pmatrix} 1 & 16 & 7 & 24 & 15 & 8 \\ 17 & 23 & 10 & 17 & 7 & 1 \\ 24 & 1 & 16 & 8 & 1 & 24 \\ 1 & 15 & 8 & 18 & 23 & 9 \\ 16 & 7 & 2 & 15 & 8 & 1 \end{pmatrix},\ B' = \begin{pmatrix} 3 & 6 & 3 & 3 & 3 & 6 & 6 & 6 & 3 & 6 \end{pmatrix}.$$

Now, let $\bar{D_1}$ and $\bar{D_2}$ be the extension codes of D_1 and D_2, respectively. By Theorem 3.9, two codes $\bar{D_1}$ and $\bar{D_2}$ have the following generator matrices, respectively.

$$\bar{G_1} = \begin{pmatrix} \infty & 0 & 1 & 2 & \cdots & p-1 \\ 0 & & & & & \\ 0 & & & G_1 & & \\ . & & & & & \\ . & & & & & \\ 24 & 16 & 16 & 16 & \cdots & 16 \end{pmatrix} \text{ and } \bar{G_2} = \begin{pmatrix} \infty & 0 & 1 & 2 & \cdots & p-1 \\ 0 & & & & & \\ 0 & & & G_2 & & \\ . & & & & & \\ . & & & & & \\ 24 & 16 & 16 & 16 & \cdots & 16 \end{pmatrix}.$$

Theorem 3.9, shows that two codes $\bar{D_1}$ and $\bar{D_2}$ are self-dual code of length 12 over the ring $R = Z_{25} + uZ_{25}$. Note that, $\mid \bar{D_i} \mid = \mid D_i \mid = 25^{12}$, for $i = 1, 2$. By Theorem 2.2, $\varphi(\bar{D_1})$

and $\varphi(\bar{D}_2)$ are self-dual code of length 24 over Z_{25}, dimension 12 and minimum Hamming weight 8.

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